Tutorial 11

(b) for $t>0$ $\quad \Phi = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right)$

$\partial_t \Phi = \underline{-\frac{n}{2}\frac{1}{t}\Phi} - \underline{\frac{|x|^2}{4t^2}\Phi}$

$\partial_i \Phi = -\frac{2x_i}{4t}\Phi = -\frac{x_i}{2t}\Phi$

$\partial_i^2 \Phi = -\frac{1}{2t}\Phi + \left(\frac{x_i}{2t}\right)^2 \Phi = -\underbrace{\frac{1}{2t}\Phi}_{n\times} + \underbrace{\frac{x_i^2}{4t^2}\Phi}_{|x|^2}$

$\Delta\Phi$

For $t=0$ need to understand limit

All derivatives go to 0

$$(c,d) \quad F_\Phi(\varphi) = \int_{\mathbb{R}}\int_{\mathbb{R}^n} \Phi(x,t)\,\varphi(x,t)\,dx\,dt = \int_0^\infty\int_{\mathbb{R}^n} \Phi\,\varphi\,dx\,dt$$

$H(\varphi)$ $\quad \parallel$

$$= \int_{\mathbb{R}^{n+1}} \Phi\,\varphi \; d(x,t)$$

prove $\Phi \in L^1_{loc}$

$$\leq \|\varphi\|_\infty \int_0^T \underbrace{\int_{\mathbb{R}^n} \Phi\,dx}_{=1 \text{ by } 4.6}\,dt = \|\varphi\|_\infty T$$

$$\int_K |\Phi|\,dx\,dt \leq \int_0^T\int_{\mathbb{R}^n} \Phi\,dx\,dt = T$$

(g) Show $\int_0^\varepsilon \int_{\mathbb{R}^n} \Phi\,dx\,dt \to 0$

$$= \int_0^\varepsilon 1\,dt = \varepsilon \to 0$$

Fundamental soln of $Lu=0$ $\qquad$ eg $\Delta u = 0$
is a distribution $\qquad$ or $(\partial_t - \Delta)u = 0$

$$L\Phi = \delta$$

Need to show

$$(\partial_t - \Delta)F_\Phi = \delta$$

$$((\partial_t - \Delta)F_\Phi)(\varphi) = (\partial_t F_\Phi)(\varphi) - (\Delta F_\Phi)(\varphi)$$
$$= F_\Phi(-\partial_t\varphi) - F_\Phi(\Delta\varphi)$$
$$= F_\Phi(-\partial_t - \Delta\varphi)$$
$$= \int_0^\infty\int_{\mathbb{R}^n} \Phi\,(-\partial_t\varphi - \Delta\varphi)\,dx\,dt$$

$$\int_\varepsilon^\infty\int_{\mathbb{R}^n} \Phi\,(-\partial_t\varphi)\,dx\,dt + \int_\varepsilon^\infty\int_{\mathbb{R}^n} \Phi\,(-\Delta\varphi)\,dx\,dt$$

$$= \left[\int_{\mathbb{R}^n} \Phi(-\varphi)\,dx\right]_{t=\varepsilon}^{t=\infty} - \int_\varepsilon^\infty\int_{\mathbb{R}^n} \partial_t\Phi(-\varphi)\,dx\,dt$$

$$+\, 0 + \int_\varepsilon^\infty\int_{\mathbb{R}^n} \Delta\Phi\,(-\varphi)\,dx\,dt$$

$$= 0 + \int_{\mathbb{R}^n} \Phi(x,\varepsilon)\varphi(x,\varepsilon)\,dx$$
$$+ \int_\varepsilon^\infty\int_{\mathbb{R}^n} \underbrace{(\partial_t\Phi - \Delta\Phi)}_{0}\,\varphi\,dx\,dt$$

$$= \int_{\mathbb{R}^n} \Phi(-x,\varepsilon)\,\underline{\underline{\varphi(x,\varepsilon)}}\,dx$$

We know Thm 4.7 that

$$\int_{\mathbb{R}^n} \Phi(y-x,\varepsilon)\,\underline{h(x)}\,dx \to h(y)$$

$$= \int_{\mathbb{R}^n} \Phi(0-x,\varepsilon)\varphi(x,\varepsilon)\,dx$$

$\varepsilon \to 0$

$$\to \varphi(0,0)$$

In total $(\partial_t - \Delta)F_\Phi(\varphi)$

$$= \int_0^\infty\int_{\mathbb{R}^n} \Phi(x,t)(-\partial_t - \Delta)\varphi\,dx\,dt$$
$$= \left(\int_0^\varepsilon\int_{\mathbb{R}^n} + \int_\varepsilon^\infty\int_{\mathbb{R}^n}\right)\Phi\,(-\partial_t - \Delta)\varphi\,dx\,dt$$

$$\to 0 + \varphi(0,0) = \delta(\varphi)$$

(a) $h \in C_b \cap L^1$ $\qquad$ vs 4.7 $\quad h \in C_b$

$$\int_{\mathbb{R}^n} |h| < \infty$$

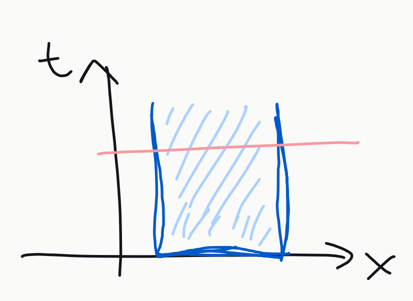
Consider $u \equiv 1$. $\qquad \int_{\mathbb{R}^n} 1 = \infty$

Prove $\sup_{x\in\mathbb{R}} |u(x,t)| \leq \underbrace{\frac{1}{(4\pi t)^{n/2}}}_{\text{decreasing in time}} \underbrace{\|h\|_{L^1}}_{\text{constant int}} \xrightarrow{t\to\infty} 0$

$$|u| \leq \int_{\mathbb{R}^n} \Phi(x-y,t)|h(y)|\,dy$$
$$= \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} \underbrace{e^{-\frac{|x-y|^2}{4t}}}_{\leq 1} |h(y)|\,dy.$$
$$= \frac{1}{(4\pi t)^{n/2}} \underbrace{\int_{\mathbb{R}^n} |h(y)|\,dy}_{\|h\|_{L^1}}$$

(b) Weak max principle.

$$\sup_{\overline{\Omega_T}} u = \sup_{\partial\Omega_T} u$$

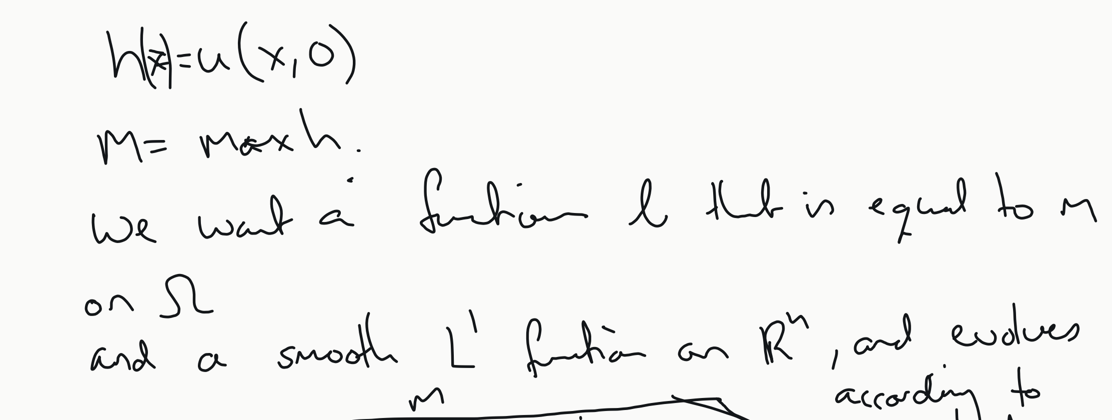


Compare u to a solution on $\mathbb{R}^n$ that starts hotter than u.

$h(x) = u(x,0)$

$M = \max h.$

We want a function ℓ that is equal to M on Ω
and a smooth L^1 function on $\mathbb{R}^n$, and evolves according to heat equation



on $\Omega \times \{0\}$ $\qquad \ell \geq u$

on $\partial\Omega \times [0,\infty)$ $\qquad \ell \geq 0 = u$

both ℓ and u solve heat eqn.

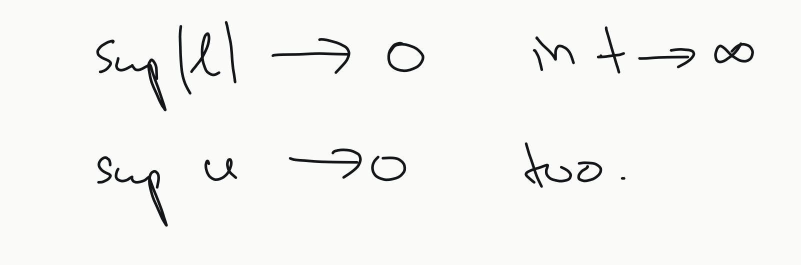
by comparison principle

$$\ell \geq u \quad \text{on all } \overline{\Omega_T}$$

Because $\ell \in L^1$ by part (a)

$\sup|\ell| \to 0 \quad$ in $t \to \infty$

$\sup u \to 0 \quad$ too.



Analogy to understand ℓ on $\mathbb{R}^n$ vs on Ω.

$\mathbb{R}^n$: Imagine a hot room with an open window that is cooling down.

Ω: Imagine an airconditioner programed to be the same temperature as the window at any time

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We can generalise:

Consider

$$(\partial_t - \Delta)u = 0 \quad \text{on } \Omega$$
$$u(x,0) = h(x)$$
$$u(x,t) = g(x) \quad \text{on } \partial\Omega$$

Consider a steady state soln v on Ω

$$(\cancel{\partial_t} - \Delta)v = 0$$
$$v(x,t) = g(x) \quad \text{on } \partial\Omega$$

v is soln to Dirichlet problem of Laplace eqn.

Consider $\tilde{u} = u - v$

$$\left.\begin{aligned}(\partial_t - \Delta)\tilde{u} &= 0\\ \tilde{u}(x,0) &= h(x) - v(x)\\ \tilde{u}(x,t) &= g - g = 0\end{aligned}\right\} \text{we solved this } \tilde{u} \to 0$$

$$u = \tilde{u} + v \to v$$