b) $\Delta h = \text{const}$

Generalise

Ansatz $u(x,t) = \sum_{k=0}^{\infty} h_k(x) t^k$

$$\partial_t u = \sum k h_k t^{k-1} = \sum_{k=0} (k+1) h_{k+1} t^k$$

$$\Delta u = \sum (\Delta h_k) t^k$$

$$0 = \partial_t u - \Delta u = \sum_{k=0} \left[(k+1)h_{k+1}(x) - \Delta h_k(x)\right] t^k$$

$$\Rightarrow h_{k+1}(x) = \frac{1}{k+1} \Delta h_k(x)$$

$$h_k(x) = \frac{1}{k!} \Delta^k h_0$$

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!} (\Delta^k h) t^k .$$

$$u(x,0) = h_0(x) + 0 + 0 + \dots$$

If $\Delta h = c$ $\quad \Delta c = 0$

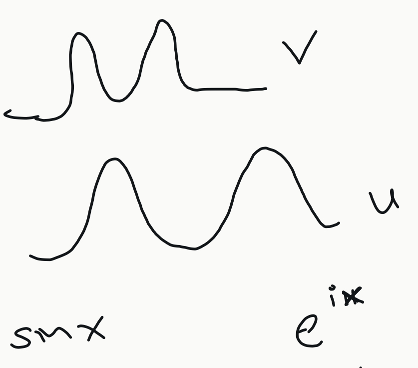
$$u = h + \frac{1}{1!} c t + \frac{1}{2!} 0 , \dots$$

(d) $v(x,t) = u(\lambda x, \lambda^2 t)$ is also a sol^n

$\lambda = 2$

$v(0,t)$ vs $v(1,t)$

$= u(0,4t)$ vs $u(2,4t)$



consider $u(x,t) = e^{-t} \sin x$ $\quad e^{ix}$

$v(x,t) = e^{-4t} \sin(2x)$ $\quad e^{2\pi i k x}$

$e^{-4\pi^2 |k|^2 t}$

$$\Phi(x, s+t) = \int_{\mathbb{R}} \Phi(y-x, s)\, \Phi(y,t)\, dy$$

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right)$$

Integral formula sol^n of Cauchy Problem

$$u(x,t) = \int_{\mathbb{R}} \Phi(y-x,t)\, h(y)\, dy$$

$$= \Phi(\cdot, t) * h$$

$$u(x, s+t) = \Phi(\cdot, s+t) * h$$

Consider this as the starting temperature of a new Cauchy problem

$$u(x, s+t) = \Phi(\cdot, s) * u(\cdot, t)$$

$$= \Phi(\cdot, s) * [\Phi(\cdot, t) * h]$$

$$= [\Phi(\cdot, s) * \Phi(\cdot, t)] * h .$$

This is a dynamical system with evolution operator $\Phi(\cdot, t) *$

$$I(0) = I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$I(k) = \int_{\mathbb{R}+ik} e^{-y^2} dy$$

$y = z + ik$ $\quad$ $y \in \mathbb{R}+ik \Leftrightarrow z \in \mathbb{R}$

$dy = dz$

$$= \int_{\mathbb{R}} e^{-(z+ik)^2} dz$$

$$\frac{\partial I}{\partial k} = \int_{\mathbb{R}} \frac{\partial}{\partial k} e^{-(z+ik)^2} dz =$$

$$= \int_{\mathbb{R}} -2(z+ik) i e^{-(z+ik)^2} dz$$

$$= i \int_{\mathbb{R}} -2(z+ik) e^{-(z+ik)^2} dz$$

$$= i \int_{\mathbb{R}} \frac{\partial}{\partial z} \left[e^{-(z+ik)^2} \right] dz$$

$$= i \left[e^{-(z+ik)^2} \right]_{z=-\infty}^{z=\infty} = 0$$

Lemma

$p(x)$ polynomial

$$\lim_{x \to \pm\infty} p(x) e^{-x^2} = 0 .$$

Proof $\lim \frac{p(x)}{e^{x^2}} = \lim \frac{p'(x)}{2x e^{x^2}} = \lim \frac{p'(x)/2x}{e^{x^2}}$

$$= \dots = \lim \frac{C}{e^{x^2}} = 0 .$$

To show that $\exp(-x^2)$ is Schwartz

$$\rho_{\alpha,\ell}(e^{-x^2}) < \infty .$$

$$\| \quad \sup_{\mathbb{R}^n} |x^{2\ell} \partial^\alpha e^{-x^2}|$$

$f = e^{-x^2}$

$f' = -2x e^{-x^2}$

$f'' = -2 e^{-x^2} + 4x^2 e^{-x^2} = (-2 + 4x^2) e^{-x^2}$

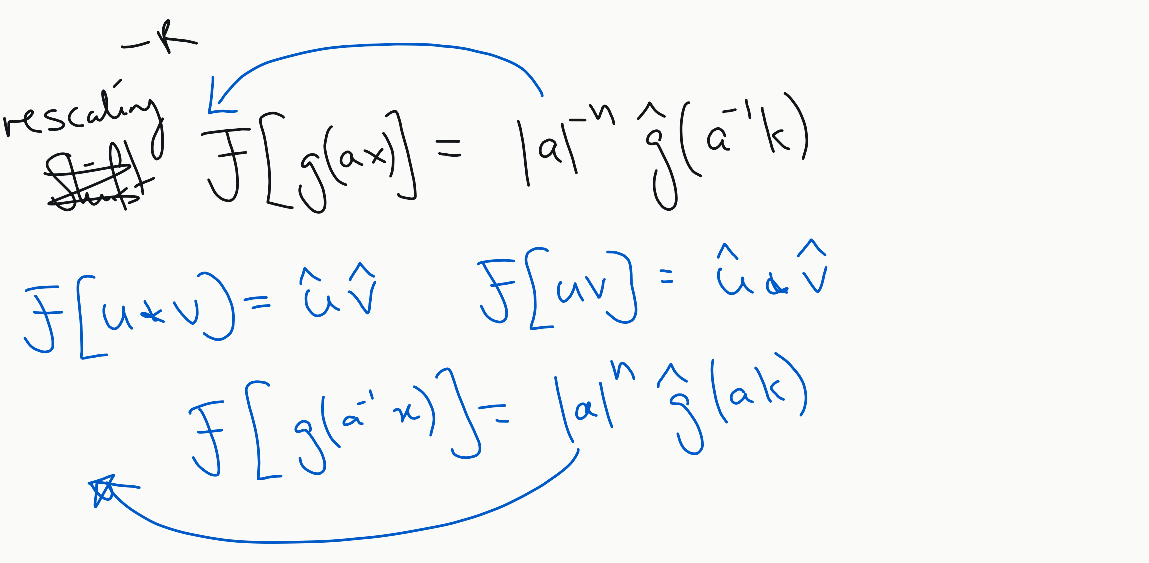
$\vdots$

$p(x) e^{-x^2}$

Need to prove $\sup_{\mathbb{R}} |p(x) e^{-x^2}| < \infty .$

$\varepsilon = 1$ $\exists R$ $|x| > R$ $\quad |p(x) e^{-x^2}| < \varepsilon = 1$

$p(x) e^{-x^2}$ is cts so bounded on $|x| \leq R$



rescaling ~~Shift~~ $\mathcal{F}[g(ax)] = |a|^{-n} \hat{g}(a^{-1}k)$

$\mathcal{F}[u * v] = \hat{u} \hat{v}$ $\quad \mathcal{F}[uv] = \hat{u} * \hat{v}$

$$\mathcal{F}[g(a^{-1}x)] = |a|^n \hat{g}(ak)$$

$$\hat{F}(\phi) = F(\hat{\phi})$$

$$\hat{\delta}(\phi) = \delta(\hat{\phi}) = \hat{\phi}(0)$$

$$= \int_{\mathbb{R}} \phi(x) e^{-2\pi i 0 \cdot x} dx$$

$$= \int_{\mathbb{R}} \phi(x) \cdot 1 \, dx$$

$$= F_1(\phi)$$

$\hat{\delta} = F_1$ $\quad$ Maximum correct

"$\hat{\delta} = 1$" $\quad$ only one possible meaning but distributions are not function

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$$\hat{1}(k) = \int_{\mathbb{R}} 1 \underbrace{e^{-2\pi i k \cdot x}}_{\text{not integrable.}} dx$$

$$= \begin{cases} k \neq 0 & \text{"morally zero"} \\ k = 0 & \int_{\mathbb{R}} 1\, dx = \infty \end{cases}$$

not defined, but similar to delta dist δ.