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Analysis III
13. Exercise: Orientation

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77. Orientable hypersurfaces defined by an equation.

Let $f : \mathbb{R}^n \to \mathbb{R}$ a smooth map, $q \in \mathbb{R}$ a point in its range so $X := f^{-1}(\{q\}) \neq \varnothing$, and f is submersive at all points $x \in X$. Show that X is an $(n-1)$-dimensional orientable submanifold.

Hint. Let ω be the volume form on $\mathbb{R}^n$ and F the gradient field of f (i.e $T_x(f)(v) = F(x) \cdot v$). Investigate $i_F\omega|X$, defined in Definition 3.11.

In Class Exercises

78. Orientable manifolds.

(a) Show that the n-dimensional sphere $\mathbb{S}^n$ is orientable by finding an oriented atlas.

Hint. For the sphere, consider the atlas that uses stereographic projection. An extra trick is also needed.

(b) Show that the Möbius band is not orientable.

Hint. This is difficult. Good luck.

(c) Let X and Y be orientable manifolds. Show that the Cartesian product $X \times Y$ is also orientable.

(d) Let X be a manifold. Show that every coordinate neighbourhood of X is orientable. More precisely, let (U, ϕ) be a chart of X with $\phi = (\phi_1, \ldots, \phi_n) : U \to \mathbb{R}^n$, and show that $\mathrm{d}\phi_1 \wedge \ldots \wedge \mathrm{d}\phi_n$ is a non-vanishing n-form on U.

(e) Prove that the tangent bundle of any manifold is orientable.

79. Integration on $\mathbb{R}$.

Consider the following integration by substitution with $f(x) = x^2$:

$$\int_{-1}^{2} 2xe^{-x^2}\, dx = \int_{-1}^{2} e^{f(x)}\, f'(x)dx = \int_{f(-1)}^{f(2)} e^u\, du = \int_{1}^{4} e^u\, du.$$

This seems very similar to Corollary 3.22, except that f is not a diffeomorphism! Also consider the substitution with $v = -x$

$$\int_{-1}^{2} e^{-x}\, dx = \int_{1}^{-2} e^u\, (-du) = \int_{-2}^{1} e^u\, du.$$

The aim of this exercise is to understand why this all works for $\mathbb{R}$.

First, extend Corollary 3.22 for orientation-reversing diffeomorphisms.

Second, define

$$\int_a^b g(x)\, dx = \operatorname{sign}(b-a) \int_{\mathbb{R}} \chi_{[a,b]}(x) g(x)\, dx.$$

Show that it obeys the rule $\int_a^b + \int_b^c = \int_a^c$.

Third, let $f : [a,b] \to \mathbb{R}$ be a continuously differentiable function. The set $(f')^{-1}[\{0\}]$ is closed, and therefore the countable union of closed intervals J_i (single points are considered as the interval $[x,x]$). Taking the endpoints of these intervals gives a partition $\{t_i\}$ of $[a,b]$ such that on each interval (t_i, t_{i+1}) the function f' is either non-zero or identically zero. Prove

$$\int_a^b g(f(x)) f'(x)\, dx = \int_{f(a)}^{f(b)} g(u)\, du.$$