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Analysis III
11. Exercise: Forms

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Preparation Exercises

66. The wedge product

In this exercise we will do some calculations with the antisymmetric algebras of $V = \mathbb{R}^3$. Let $\{e_i\}$ be a basis of V and $\{\alpha_i\}$ the corresponding dual basis of V'.

(a) Show that $\beta = \alpha_1 \otimes \alpha_2 - \alpha_2 \otimes \alpha_1$ belongs to $\bigwedge^2 V'$.

(b) Consider the antisymmeterising operation $\mathcal{A}$ defined in the proof of Theorem 3.4. Compute $\mathcal{A}^1(\alpha_3) \in \bigwedge^1 V'$ and $\mathcal{A}^2(\alpha_1 \otimes \alpha_2) \in \bigwedge^2 V'$.

(c) Compute $\langle \beta, e_1 \otimes e_3 \rangle = \beta(e_1, e_3)$ using the definition of tensors as linear maps and also using the formula on page 71 of the script (notice that $\beta = \alpha_1 \wedge \alpha_2$).

(d) Compute the wedge product $\beta \wedge \alpha_3$.

(e) In the proof of Theorem 3.4, it is proved that the wedge products span the space of antisymmetric tensors by a dimension count argument. Given an antisymmetric tensor, can you find an algorithm to write it as a sum of wedge products of basis elements?

In Class Exercises

67. Dual 1-forms to a vector field (using dot product).

Let $F \in \mathrm{Vec}^\infty(\mathbb{R}^3)$ be a smooth vector field on $\mathbb{R}^3$ that is nowhere vanishing. Find a tensor field α of $T'\mathbb{R}^3$ (a 1-differential form on $\mathbb{R}^3$), so that the kernel of α at every point is orthogonal to F. Orthogonal means using the dot product of $\mathbb{R}^3$

68. Local representations of tensor fields.

Let X be an n-dimensional smooth manifold and f a tensor field in $T^q_p X$. To simplify the notation, we will only consider the case $p = 0, q = 2$, but for other tensor spaces everything holds completely analogously. Further, let (U, ϕ) be a chart of X and denote the components of ϕ by $\phi_1, \dots, \phi_n : U \to \mathbb{R}$. These induce 1-forms $\alpha_k := \mathrm{d}\phi_k$ in $T^1_0 U \subset T^1_0 X$. By definitions these 1-forms act as $\alpha_k(x)(v) = T_x(\phi_k)(v)$ for every $x \in U$ und $v \in T_x X$.

General Hint. It will be useful to consider the (local) vector fields E_k on U

$$E_k(x) = T_x(\phi)^{-1}(e_k).$$

These are often called the coordinate vector fields.

(a) Show there exist functions $f_{k,l} : U \to \mathbb{R}$, so that

$$f|U = \sum_{k,l=1}^{n} f_{k,l} \cdot \alpha_k \otimes \alpha_l.$$

More precisely, we mean that for all $x \in U$ and $v, w \in T_x X$

$$f(x)(v,w) = \sum_{k,l=1}^{n} f_{k,l}(x)\, \alpha_k(x)(v)\, \alpha_l(x)(w).$$

Show further that these functions are unique. This way of describing f is called representing f in local coordinates. The functions $f_{k,l}$ are called the coefficient functions.

(b) Show that $f|U$ is smooth exactly when the functions $f_{k,l}$ are all smooth.

(c) Show that f is a 2-form exactly when the coefficients are antisymmetric: $f_{k,l} = -f_{l,k}$.

69. Closed and exact differential forms.

A p-form ω on a manifold X is called closed if $\mathrm{d}\omega = 0$, and it is called exact if there is a $(p-1)$-form θ on X with $\omega = \mathrm{d}\theta$.

Consider $X = \mathbb{R}^3$ and let $x, y, z : \mathbb{R}^3 \to \mathbb{R}$ be the usual coordinate functions. Investigate whether the following forms are closed and or exact.

(a) $\omega = yz\,\mathrm{d}x + xz\,\mathrm{d}y + xy\,\mathrm{d}z$

(b) $\omega = x\,\mathrm{d}x + x^2y^2\,\mathrm{d}y + yz\,\mathrm{d}z$

(c) $\omega = 2xy^2\,\mathrm{d}x \wedge \mathrm{d}y + z\,\mathrm{d}y \wedge \mathrm{d}z$