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Analysis III

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9. Exercise: Flows and Integral Curves

Preparation Exercises

52. Examples of integral curves and flows.

Let F be a smooth vector field on $\mathbb{R}^2$ given by

$$F(x, y) = (-y, x)$$

(a) Find the maximal integral curves of F.

(b) Write down the maximal flow of F.

(c) Consider the restriction of F to $\mathbb{S}^1$. What are the integral curves and maximal flow?

53. A Flow on $\mathbb{S}^2$.

Consider the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. Define a vector field $\mathbb{F} : \mathbb{S}^2 \to \mathbb{R}^3$ by

$$F(x, y, z) = (-y, x, 0).$$

(a) Show that F is a vector field on S^2 (using the identification that comes from the inclusion map $\iota : \mathbb{S}^2 \to \mathbb{R}^3$).

(b) What are the integral curves of F?

(c) Determine the maximal flow ψ of F.

(d) Let $M := \mathbb{S}^2 \setminus \{(1, 0, 0)\}$. Find an open neighbourhood W of $\{0\} \times M$ in $\mathbb{R} \times M$ so that $\psi|W$ is a flow on M. Is $\psi|W$ a global flow on M?

In Class Exercises

54. A non-integrable subbundle

Consider the sections f_1, f_2, f_3 on $T\mathbb{S}^3$ from exercise 37 (b). Show that for any $i, j \in \{1, 2, 3\}$ with $i \neq j$ the sub-vectorbundle of $T\mathbb{S}^3$ spanned by $< f_i, f_j >$ is not integrable, which means it doesn't fulfill the condition from Frobenius theorem (2.23).

55. An integrable subbundle

(a) Let X be a manifold and $F \in \mathrm{Vec}(X)$ be a vectorfield. Let $f \in C^1(X, \mathbb{R})$. Show that

$$\theta_{fF}(g) = f\theta_F(g) \quad \text{for all } g \in C^1(X, \mathbb{R}).$$

(b) Now consider F to be a non-zero vector field. Show that sub-vectorbundle spanned by F is integrable, which means it does fulfill the condition from Frobenius theorem.

56. An example of an non-complete vector field.

Let

$$W := \{\, (t,(x,y)) \in \mathbb{R} \times \mathbb{R}^2 \,|\, 2\,(x^2+y^2) \cdot t < 1 \,\}$$

and

$$\psi : W \to \mathbb{R}^2,\ (t,(x,y)) \mapsto \frac{1}{\sqrt{1-2\,(x^2+y^2)\cdot t}} \cdot (x,y).$$

(a) Show that ψ is a flow on $\mathbb{R}^2$.

(b) Determine the corresponding vector field $F \in \mathrm{Vec}^\infty(\mathbb{R}^2)$.

(c) Explain why ψ is the maximal flow of F, and why F is not a complete vector field.

57. Commuting flows.

Let $a,b,c \in \mathbb{R}$ be constants and the vector fields $F,G \in \mathrm{Vec}^\infty(\mathbb{R}^3)$ be given by

$$F(x_1,x_2,x_3) = (1, x_3, -x_2) \quad \text{and} \quad G(x_1,x_2,x_3) = (a,b,c)\,.$$

(a) Determine the flows ψ_F and ψ_G induced by F and G respectively, and determine for which values of a,b,c the flows commute with one another: i.e. for all $t,s \in \mathbb{R}$

$$\psi_F(t,\psi_G(s,x)) = \psi_G(s,\psi_F(t,x)).$$

(b) Calculate $[F,G]$, and determine for which values of a,b,c the Lie bracket is zero, $[F,G]=0$.

58. A trichotomy of integral curves.

Let X be a manifold, F a smooth vector field on X, $x_0 \in X$, and $\gamma : J \to X$ the maximal integral curve of F with $\gamma(0) = x_0$.

(a) Show there is a trichotomy: either γ is constant, or γ is injective, or γ is periodic, and these are mutually exclusive. Periodic means that $J = \mathbb{R}$, γ is non-constant, and there is a number $p > 0$ so that

$$\gamma(t+p) = \gamma(t) \quad \text{for all } \ t \in \mathbb{R}.$$

This number p is called a *period* of γ. It is not unique; for example if p is a period, so is $2p$.

Hint: Assume that γ is not constant or injective, and try to show that it is periodic.

(b) Show γ is constant exactly when $F(x_0) = 0$.

(c) Suppose that γ is periodic. Show that there is a *minimal period* $p_0 > 0$: that means p_0 is a period of γ and there are no other periods in the interval $0 < p < p_0$.

Hint: Prove this by contradiction.

(d) Suppose that γ is periodic. Show that any period is a multiple of the minimal period.

(e) Suppose that γ is periodic. Show that $\gamma|_{[0,p_0)}$ is injective and the map $f : \mathbb{S}^1 \to X$ defined by

$$f(\cos(\theta), \sin(\theta)) = \gamma\left(\frac{p_0}{2\pi} \cdot \theta\right) \quad \text{for all } \theta \in \mathbb{R}$$

is an embedding with $f[\mathbb{S}^1] = \gamma[\mathbb{R}]$. It follows that that the image $\gamma[\mathbb{R}]$ is a submanifold of X.

Hint: Constant Rank Theorem.

Additional Exercises

59. The integral curves of vector fields with the form $\boldsymbol{\lambda F}$.

Let X be a manifold, $F \in \mathrm{Vec}^\infty(X)$ a vector field, $\lambda \in C^\infty(X, \mathbb{R})$ a smooth function, $G := \lambda F \in \mathrm{Vec}^\infty(X)$ the rescaling of F, and $p_0 \in X$ a point.

Suppose that $\alpha : I \to X$ is an integral curve of F with $\alpha(0) = p_0$ and that $f : J \to I$ is a solution to the initial value problem

$$f'(t) = \lambda(\alpha(f(t))) \quad \text{with} \quad f(0) = 0.$$

Show then that $\beta := \alpha \circ f : J \to X$ is an integral curve of G with $0 \in J$ and $\beta(0) = p_0$.

Moreover, show that every integral curve of G can be obtained in this way.

60. Integral curves on the torus.

For each $a > 0$ let

$$F_a : \mathbb{S}^1 \to T\mathbb{S}^1, \ (x_0, x_1) \mapsto \big(a(-x_1, x_0), (x_0, x_1)\big)$$

be a non-vanishing smooth vector field with the maximal integral curve $\gamma_a : \mathbb{R} \to \mathbb{S}^1$ with $\gamma_a(0) = (1, 0)$.

Next we consider the 2-dimensional manifold $\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$. This subset of $\mathbb{R}^2 \times \mathbb{R}^2$ is a torus, a doughnut (donut). For constants $a, b > 0$ we define the vector field

$$G_{a,b} : \mathbb{T}^2 \to T\mathbb{T}^2, \ \big((x_1, y_1), (x_2, y_2)\big) \mapsto \big(F_a(x_1, y_1)\, ,\, F_b(x_2, y_2)\big)\ .$$

(a) Prove that the curve

$$\eta_{a,b} : \mathbb{R} \to \mathbb{T}^2 = T\mathbb{S}^1 \times T\mathbb{S}^1,\ t \mapsto \big(\gamma_a(t), \gamma_b(t)\big)$$

is the maximal integral curve of $G_{a,b}$ with $\eta_{a,b}(0) = \big((1,0),(1,0)\big) \in \mathbb{T}^2$.

(b) Suppose $\frac{a}{b} \in \mathbb{Q}$. Show that $\eta_{a,b}$ is periodic and determine the minimal period.

The image is a submanifold called a *torus knot*.

(c) Suppose $\frac{a}{b} \in \mathbb{R} \setminus \mathbb{Q}$. Show that $\eta_{a,b}$ is injective, but that it is not an embedding.

Remark. In this case, the image $\eta_{a,b}[\mathbb{R}]$ is in fact dense in $\mathbb{T}^2$.

61. Aligning coordinates with a vector field.

Again let X be a manifold. Let $n := \dim(X)$ be its dimension, $x_0 \in X$ a point, and $F \in \mathrm{Vec}^\infty(X)$ a vector field with $F(x_0) \neq 0$. Show that there is a chart (U, ϕ) containing $x_0 \in U$ such that

$$T_x(\phi)^{-1}(e_1) = F(x) \quad \text{for all } x \in U.$$

Hint: Let ψ be the maximal flow of F. Then we know that ψ is defined on $(-\varepsilon, \varepsilon) \times U'$ for some $\varepsilon > 0$ and neighbourhood $U' \ni x_0$. Next choose an $(n-1)$-dimensional submanifold S of U' with $x_0 \in S$ and $F(x_0) \notin T_{x_0}S$ (explain why there must exists such an S). Finally, apply the inverse function theorem to ψ.