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Analysis III
8. Exercise: Vector Fields

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Preparation Exercises

47. Coordinate vector fields.

Let $\phi : U \to \mathbb{R}^n$ be a chart of X for an open set $U \subset X$. Then consider the vector field $F_i : U \to TU$ with

$$F_i(x) = T_x(\phi)^{-1}(e_i) \in T_xU \ ,$$

for $i \in \{1, \ldots, n\}$ and where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n$ is the i-th standard unit vector of $\mathbb{R}^n$. This is called a coordinate vector field.

(a) Show that these are vector fields $F_i \in \mathrm{Vec}^\infty(U)$.

(b) Show that any other vector field F on U can be written

$$F(x) = \sum_i a_i(x) F_i(x)$$

for smooth functions $a_i : U \to \mathbb{R}$.

48. Vector fields and derivations.

(a) For a vector field F on X, describe the difference and relationship between the derivation θ_F defined by Theorem 2.2 and D_v described by Theorem 1.40.

(b) What is the derivation that corresponds to a coordinate vector field?

(c) Suppose that $F = \sum_i a_i(x) F_i(x)$ as in the previous exercise. Show that $\Theta_F = \sum_i a_i \Theta_{F_i}$.

In Class Exercises

49. The Lie bracket in $\mathbb{R}^n$.

The Lie bracket is the name of the operation on vector fields defined in Corollary 2.3. Let us focus on $\mathbb{R}^n$. The tangent bundle is trivial $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ and we can write a vector field as $F : \mathbb{R}^n \to \mathbb{R}^n$ (technically we should write $F(x) = (\tilde{F}(x), x)$, but the tildes are annoying).

(a) How can we calculate $\theta_F(f)$ for some function $f : \mathbb{R}^n \to \mathbb{R}$?

(b) Let $F, G : \mathbb{R}^n \to \mathbb{R}^n$ be two vector fields on $\mathbb{R}^n$. Show

$$[F, G](x) = JG(x)\, F(x) - JF(x)\, G(x) \ .$$

(c) Consider the three vector fields on $\mathbb{R}^4$ (we have seen these already in the exercise about $T\mathbb{S}^3$):

$$
\begin{aligned}
F(x_1,x_2,x_3,x_4) &:= (-x_2,x_1,x_4,-x_3)\ ,\\
G(x_1,x_2,x_3,x_4) &:= (-x_3,-x_4,x_1,x_2)\\
\text{and}\quad H(x_1,x_2,x_3,x_4) &:= (-x_4,x_3,-x_2,x_1)\ .
\end{aligned}
$$

(i) Calculate $[F,G]$, $[G,H]$ und $[F,H]$.

(ii) For these three fields, check that the *Jacobi identity* holds:

$$[F,[G,H]] = [[F,G],H] + [G,[F,H]]\ .$$

50. The computation of the Lie Bracket for submanifolds of $\mathbb{R}^n$.

Let $X \subset \mathbb{R}^n$ be a submanifold of $\mathbb{R}^n$ and $F, G \in \mathrm{Vec}^\infty(X)$. With the help of Theorem 2.22(iii),(iv) devise a formula to compute $[F,G]$. Prove your formula.

Additional Exercises

51. Properties of the Lie bracket. Let X be an n-dimensional manifold.

(a) Show: the Lie bracket has the following properties for all vector fields $F,G,H \in \mathrm{Vec}^\infty(X)$ and scalars $a \in \mathbb{R}$.

(i) $\mathbb{R}$-linear: $[aF,G] = a[F,G]$.

(ii) anti-symmetric: $[F,G] = -[G,F]$.

(iii) Jacobi identity: $[F,[G,H]] + [G,[H,F]] + [H,[F,G]] = 0$.

Hint: The pairing $F \to \theta_F$ is injective (and for smooth vector fields and derivations it is bijective), so it is enough to show equality for the corresponding derivations. Eg. to show $[aF,G] = a[F,G]$ you can show $\theta_{[aF,G]} = \theta_{a[F,G]}$.

(b) Show that coordinate vector fields commute: $[F_i, F_j] = 0$ for every i, j.