Martin Schmidt Nicolas A. Hasse

Analysis III 7. Exercise: Vector Bundles II

Preparation Exercises

41. A non-trivial line bundle over \mathbb{R}/\mathbb{Z} .

Consider the lines $L_x = \mathbb{R}(\cos \pi x, \sin \pi x) \subset \mathbb{R}^2$. Notice that $L_{x+1} = L_x$.

- (a) By writing M := {(v, [x]) ∈ ℝ² × ℝ/ℤ | v ∈ L_x} locally as the level set of functions F : ℝ² × U_x → ℝ, prove it is a submanifold.
 Hint. How can you write the line L_x as the zero set of a function?
- (b) Show M is a vector bundle over ℝ/ℤ. It is called the Möbius band or Möbius bundle. Hint. Can you find non-vanishing local sections?
- (c) Compute the cocycle $g_{U_{0.5},U_0}$.
- (d) Prove that $(M, \mathbb{R}/\mathbb{Z}, \pi)$ is a non-trivial bundle.

Hint. Prove there are no non-vanishing global sections.

In Class Exercises

42. Isomorphism of bundles given as cocycles.

We have seen that cocycles are a convenient way to define a bundle. Suppose that we have two bundles E and E' over B given by cocycles g and g' on the same covering of B. Suppose that there exist smooth functions $h_U: U \to \operatorname{GL}(F)$ for each open set U in the cover such that

$$g'_{V,U}(b) = h_V(b) g_{V,U}(b) h_U(b)^{-1}.$$

Show that the bundles E and E' are isomorphic.

43. Direct sum of two Möbius bundles over \mathbb{R}/\mathbb{Z} .

Consider the Möbius bundle M over \mathbb{R}/\mathbb{Z} . Prove that $M \oplus M$ is the trivial rank 2 bundle over \mathbb{R}/\mathbb{Z} .

44. Classification of line bundles over \mathbb{R}/\mathbb{Z} .

In this exercise we will classify all line bundles over \mathbb{R}/\mathbb{Z} . Let *E* be a line bundle over \mathbb{R}/\mathbb{Z} .

- (a) Use a previous exercise to argue that $\pi^{-1}[U_0]$ and $\pi^{-1}[U_{0.5}]$ must be trivial bundles.
- (b) Explain why $GL(\mathbb{R}) = \mathbb{R}^{\times}$.

(c) Because E trivialisations over the cover $\mathcal{U} = \{U_0, U_{0.5}\}$, we can describe E over by a single cocycle $g = g_{U_{0.5},U_0} : U_0 \cap U_{0.5} \to \mathbb{R}^{\times}$. Let $\{\varphi_U : U \to [0,1]\}$ be a partition of unity for \mathcal{U} . Define $h_U : U \to \mathbb{R}^+$ by

$$h_U([x]) = \left(\sum_{V \in \mathcal{U}} \varphi_V([x]) g_{V,U}([x])^2\right)^{1/2}.$$

For example,

$$h_{U_0}([x]) = \left(\varphi_{U_0}([x])g_{U_0,U_0}([x])^2 + \varphi_{U_{0.5}}([x])g_{U_{0.5},U_0}(x)^2\right)^{1/2}$$

= $\left(\varphi_{U_0}([x]) + \varphi_{U_{0.5}}([x])g([x])^2\right)^{1/2}$.

Why is h_{U_0} well-defined at [x] = [0] even though $g_{U_{0.5},U_0}([x])$ is not defined there? Show, using the cocycle properties that $h_{U_{0.5}}h_{U_0}^{-1} = |g|^{-1}$.

- (d) Hence show that E is isomorphic to a bundle whose cocycle has $|g'_{U_{0.5},U_0}(x)| = 1$.
- (e) Show that E is isomorphic to either the trivial bundle $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$ or the Möbius bundle.

Additional Exercises

45. Triviality of the homomorphism bundle.

Let (E, B, π) and (E', B, π') be two vector bundles over a base manifold B. Consider the homomorphism bundle (Hom $(E, E'), B, \pi''$). People often say "Hom-bundle" for short. In parts (b) and (c) there are two methods of proof: try to find a non-vanishing sections or examine the cocycles.

- (a) What is the rank of Hom(E, E').
- (b) Show that when E and E' are trivial bundles, then so too is Hom(E, E').
- (c) Prove or disprove: Hom(E, E') is trivial, then the bundles E and E' must be trivial. Hint: Examine the Möbius bundle M.

46. The dual bundle of a vector bundle.

Let (E, B, π) be a vector bundle over a manifold B, with fibre $F = \mathbb{R}^n$. Further, let \mathcal{U} be an open cover of B so that π trivialises over every set $U \in \mathcal{U}$. Denote the cocycles of π with respect to this cover by $g_{V,U} : U \cap V \to \operatorname{GL}(\mathbb{R}^n)$.

Show that the dual bundle $(\tilde{E}, B, \tilde{\pi})$ to π is described over \mathcal{U} by the cocycle $(\tilde{g}_{V,U})_{V,U \in \mathcal{U}}$ with

$$\tilde{g}_{V,U}: U \cap V \to \operatorname{GL}(\mathbb{R}^n), \ x \mapsto (g_{V,U}(x)^T)^{-1}.$$