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Analysis III
4. Exercise: Tangent Space

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In Analysis II we learn that the derivative of a function $F = (F_1, \dots, F_n) : \mathbb{R}^m \to \mathbb{R}^n$ at a point a is a linear map, an $n \times m$ matrix,

$$J_a F := \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(a) & \dots & \frac{\partial F_1}{\partial x_m}(a) \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1}(a) & \dots & \frac{\partial F_n}{\partial x_m}(a) \end{pmatrix}.$$

I call this matrix the Jacobian and denote it with the letter J. This is so we do not confuse it with derivations, which often use the letter D. Martin uses a prime $F'(a)$. All three $J, D, '$ are common. When the matrix is a single column we identify it with the vector. ∇ is also a common notation in this case. When it is 1×1 we identify it with the real number. We have the chain rule

$$J_a(F \circ G) = J_{G(a)}F \circ J_a G.$$

When thinking of these as matrices and vectors, we may omit the $\circ$ and use juxtaposition to represent matrix multiplication.

Preparation Exercises

20. Tangency for curves.

In Definition 1.32 we define the concept of tangency for smooth maps f_1, f_2 at a point $x = f_1(t) = f_2(t)$. In this exercise we examine this concept for curves. A curve through $x \in X$ is a smooth map $\alpha : (-\varepsilon, \varepsilon) \subset \mathbb{R} \to X$ such that $\alpha(0) = x$. Another way to say Definition 1.33 is that the tangent space $T_x X$ is the set of curves through x with the equivalence relation of tangency.

(a) Give the definition for two curves α, β through x to be tangential at x.

Let ϕ be a chart that contains x. Show that these curves are tangential at x if and only if $J_0(\phi \circ \alpha) = J_0(\phi \circ \beta) \in \mathbb{R}^n$.

(b) Choose $w \in \mathbb{R}^n$. Show that $\alpha_w(t) := \phi^{-1}(tw + \phi(x))$ is a curve through x.

(c) Show that every curve through x is equivalent to α_w for some w. This shows that $T_x X = \{[\alpha_w] \mid w \in \mathbb{R}^n\}$. This correspondence is called 'writing a tangent vector in coordinates'.

21. Derivations at a point.

We refer to Theorem 1.40 in the script and the explanation that proceeds it.

(a) Consider $D : C^\infty(X, \mathbb{R}) \to \mathbb{R}$ and a point $x \in X$. Recall the definition that D is a derivation at x.

(b) Prove that the set of derivations at x is a vector space.

(c) Let α be a curve through x. Show that $D_\alpha : C^\infty(X,\mathbb{R}) \to \mathbb{R}$ given by $D_\alpha(f) = (f \circ \alpha)'(0)$ is a derivation at x.

(d) Show that evaluating a partial derivative of a function with respect to a chart ϕ (see Exercise 14(e)) is a derivation at x. That is,

$$f \mapsto \frac{\partial f}{\partial \phi_i}(x) = \frac{\partial (f \circ \phi^{-1})}{\partial y_i}(\phi(x)).$$

In Class Exercises

22. The tangent map.

Let X, Y be manifolds and $f : X \to Y$. The map $T_x(f) : T_x X \to T_{f(x)} Y$ is called the tangent map is defined in Definition 1.35. It is also called the push-forward map or the differential.

(a) Prove that if α and β are curves through x that are tangential, then $T_x(f)(\alpha)$ is tangential to $T_x(f)(\beta)$ at $f(x)$. This shows that $T_x(f)$ is indeed well-defined between tangent spaces.

(b) Suppose that $Y = \mathbb{R}^m$. Using the canonical identification $T_y\mathbb{R}^m = \mathbb{R}^m$ show how to identify $T_x(f)(\alpha)$ with a vector in $\mathbb{R}^m$. How does this relate to Exercise 20(a)?

(c) Let X be connected. Show that f is constant if and only if $T_x(f) = 0$ for all $x \in X$.

23. Examples of tangent vectors.

(a) Let $\alpha, \beta : \mathbb{R} \to \mathbb{S}^1$ be given by $\alpha(t) = (\sin t, \cos t)$ and $\beta(t) = (\sin t^2, \cos t^2)$. Do these curves through $(0,1)$ give the same tangent vector in $T_{(0,1)}\mathbb{S}^1$?

(b) Write α in coordinates with respect to ϕ_N and ϕ_S.

(c) How do you transform a vector written in coordinates with respect to one chart into another chart?

(d) Let $\iota : \mathbb{S}^1 \to \mathbb{R}^2$ be the map $\iota(x_1, x_2) = (x_1, x_2)$. This is called the inclusion map. Let v be a tangent vector in $T_x\mathbb{S}^1$. Show that $w := T_x(\iota)(v) \in \mathbb{R}^2$ is perpendicular to x.

Conversely, choose any $w \in \mathbb{R}^2$ with $w \cdot x = 0$ and set $\alpha(t) = (\cos|w|t)x + (\sin|w|t)\widehat{w}$. Show that $w = T_x(\iota)([\alpha])$. *(2 Points)*

Hence we make the identification

$$T_x\mathbb{S}^1 = \{\, w \in \mathbb{R}^2 \mid w \cdot x = 0 \,\}\,.$$

(e) Recall the maps $f : \mathbb{S}^1 \to \mathbb{R}$ and $A : \mathbb{S}^1 \to \mathbb{S}^1$ from Exercise 16. Write $T_{(1,0)}(f)(\alpha)$ and $T_{(1,0)}(\iota \circ A)(\alpha)$ using natural identifications. Interpret these results.

24. The tangent space as a vector space.

We know that the tangent space T_xX is the set of equivalence classes of curves through x. We want this to be a vector space, but one cannot add curves to one another.

As in Theorem 1.36 let ϕ be a chart centered on x, $\phi(x) = 0$. Then because ϕ is a homeomorphism, the induced map $\Phi := T_x(\phi)$ is a bijection from $V := T_xX$ to $T_{\phi(x)}\mathbb{R}^n = \mathbb{R}^n$. We give V the structure of a vector space which makes Φ an isomorphism. Explicitly

$$[\alpha] + [\beta] = \Phi^{-1}\Big(\Phi([\alpha]) + \Phi([\beta])\Big),$$
$$a[\alpha] = \Phi^{-1}\Big(a\Phi([\alpha])\Big).$$

The question is, does the vector space structure depend on the choice of chart ϕ? If we add two vectors according to one chart, do we get the same answer to when we add them according to another chart?

Prove directly that the vector space structure on the tangent space does not depend on the choice of chart (Theorem 1.36(i)).

Additional Exercises

25. Tangent vectors and Derivations at a point.

Using the results already developed in this exercise sheet, give your own proof of Theorem 1.40. That is, prove that $[\alpha] \to D_\alpha$ is a well-defined bijection and that it preserves the vector space structure.

Terminology

f und g berühren sich $=$ f and g are tangential.