Introduction to Partial Differential Equations
Exercise sheet 13

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38. Distinct characteristics

(a) Show that the a smooth function $u = u(\zeta, \eta) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a solution to $\partial_{\zeta\eta} u = 0$ exactly when it is of the form $u(\zeta, \eta) = F(\zeta) + G(\eta)$, for smooth functions $F, G : \mathbb{R} \to \mathbb{R}$.

(2 points)

(b) Under the parameterisation $\zeta = x + t, \eta = x - t$, show that u obeys the one dimensional wave equation $(\partial_{tt} - \partial_{xx})u = 0$ exactly when it solves the PDE in (a). *(3 points)*

(c) From parts (a) and (b), derive D'Alembert's formula. *(2 points)*

39. Faster!

How should you modify D'Alembert's formula for this situation?

$$\begin{cases} \partial_{tt} u - a^2 \partial_{xx} u = 0 \\ u(x, 0) = g(x) \\ \partial_t u(x, 0) = h(x), \end{cases}$$

Solve this for the initial data $a = 2$, $g(x) = \sin(x)$ and $h(x) = 1$. *(5 points)*

40. Plane Waves

Suppose that $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a solution to the following modified wave equation:

$$\frac{\partial^2 u}{\partial t^2} - \sum_{j=1}^{n} c_j^2 \frac{\partial^2 u}{\partial x_j^2} = 0 \,, \tag{$*$}$$

where $c_1, \ldots, c_n > 0$ are constants.

(a) Let $\alpha \in \mathbb{R}^n$ be a unit vector $\|\alpha\| = 1$, $\mu \in \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ a twice continuously differentiable function. Show that

$$u(x, t) := F(\alpha \cdot x - \mu t)$$

is a solution of $(*)$ exactly when

$$\mu^2 = \sum_{j=1}^{n} \alpha_j^2 c_j^2$$

or F is linear. Solutions of $(*)$ with this form are called *plane waves.* *(2 points)*

(b) For the solutions in (a), examine whether the following property holds for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$:

$$u(x, t) = u(x - \mu t \alpha, 0).$$

Interpret this equation in terms of direction and speed. *(3 points)*

41. Electromagnetic Waves

In physics, electrical and magnetic fields are modelled as time-dependent vector fields, which mathematically are smooth functions $E, B : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$. Through a series of experiments in the 18th and 19th centuries, the existence and properties of these fields were discovered. Importantly, it was discovered that the two phenomena were connected (both magnets and static electricity had been known since antiquity). In 1861 James Clerk Maxwell published a series of papers summarising electromagnetic theory, including a collection of 20 differential equations. Over time these were further reduced to the following four (by Heaviside 1884 using vector notation), called *Maxwell's Equations*:

$$\nabla \cdot E = \frac{1}{\varepsilon_0}\rho \qquad \nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \cdot B = 0 \qquad \nabla \times B = \mu_0 J + \varepsilon_0 \mu_0 \frac{\partial E}{\partial t}.$$

As is usual, the ∇ operator acts on the spatial coordinates x, and the $\times$ denotes the cross product of $\mathbb{R}^3$. The constants ε_0, the electrical permittivity, and μ_0, the magnetic permeability, are approximately $\varepsilon_0 \approx 8,854 \cdot 10^{-12} \frac{\mathrm{A \cdot s}}{\mathrm{V \cdot m}}$ and $\mu_0 \approx 1,257 \cdot 10^{-6} \frac{\mathrm{V \cdot s}}{\mathrm{A \cdot m}}$ (V=Volt, s=Seconds, A=Ampere and m=Metre) in a vacuum. Electrical charges are included via the charge density ρ and electric currents are the movements of charges, $J := v\rho$ for a velocity field v.

The two equations with divergence were formulated by Gauss, based on known inverse-square force laws, the curl of the electric field is due to Faraday, and the curl of the magnetic field is due to Ampère. The last term in Ampère's law that has the time-derivative of the electrical field was an addition of Maxwell. With this correction, he was able to derive the equations for electromagnetic waves, as you will now do.

(a) Let E und B be solutions to Maxwell's equations in the absence of electric charges, $\rho = 0, J = 0$. Show that they each satisfy a modified wave equation (Question 40). You may use without proof the identity $\nabla \times (\nabla \times f) = \nabla(\nabla \cdot f) - \triangle f$ for smooth functions $f : \mathbb{R}^3 \to \mathbb{R}^3$. *(3 points)*

(b) Predict the speed of these waves. *(2 Bonus Points)*

(c) Argue that Ampère's law in its original form $\nabla \times B = \mu_0 J$ violates the conservation of charge ρ under some conditions. Refer to Exercise Sheet 5 for the definition of a conservation law. Thereby derive Maxwell's additional term. *(3 Bonus Points)*