Introduction to Partial Differential Equations
Exercise sheet 12

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35. Some like it hot

Find the solution $u : (0, \pi) \times \mathbb{R}^+ \to \mathbb{R}$ of the initial and boundary value problem: *(6 points)*

$$\begin{cases} \dot{u} - 7\partial_{xx} u = 0 & \text{for } x \in (0,\pi),\ t > 0 \\ u(0,t) = u(\pi,t) = 0 & \text{for } t > 0 \\ u(x,0) = 3\sin(2x) - 6\sin(5x) & \text{for } x \in (0,\pi). \end{cases}$$

36. Out of the frying pan, into the fire

Find the solution $u : (0, \pi) \times \mathbb{R}^+ \to \mathbb{R}$ of the initial and boundary value problem:

$$\begin{cases} \dot{u} - \partial_{xx} u = 0 & \text{for } x \in (0,\pi),\ t > 0 \\ u(0,t) = u(\pi,t) = 0 & \text{for } t > 0 \\ u(x,0) = x^2(\pi - x) & \text{for } x \in (0,\pi). \end{cases}$$

Further, show that your solution obeys $\displaystyle\int_0^\pi u(x,t)\,\mathrm{d}x = 8 \sum_{k \text{ odd}} \frac{1}{k^4} e^{-k^2 t}$. *(7 points)*

37. Method of images for the heat kernel on $[0, 1]$

(a) Show the following formula for theta functions

$$\Theta(\tfrac{z}{2}, \pi i t) = 1 + \sum_{k=1}^{\infty} e^{-\pi^2 t k^2} 2\cos(\pi k z),$$

and therefore that

$$H_{[0,1]}(x,y,t) = \frac{1}{2}\Theta(\tfrac{x-y}{2}, \pi i t) - \frac{1}{2}\Theta(\tfrac{x+y}{2}, \pi i t)$$

as claimed in the script. *(2 points)*

(b) Let $\mathcal{A}$ be the space of all continuous functions on $\mathbb{R}$ with the following properties:

$$f(n+x) = \begin{cases} f(x) & \text{for even } n \in 2\mathbb{Z} \text{ and } x \in \mathbb{R} \\ -f(1-x) & \text{for odd } n \in 2\mathbb{Z}+1 \text{ and } x \in \mathbb{R}. \end{cases}$$

Show that the functions in $\mathcal{A}$ vanish at $\mathbb{Z}$ and that $\mathcal{A}$ contains all continuous odd and periodic functions with period 2. *(1 point)*

(c) Show that for any Schwartz function f on $\mathbb{R}$ the following series converges to a smooth functions $\tilde{f}$ in $\mathcal{A}$: *(2 points)*

$$\tilde{f}(x) = \sum_{n\in\mathbb{Z}} f(2n + x) - \sum_{n\in\mathbb{Z}} f(2n - x).$$

(d) Conclude that the following sum is a heat kernel of $[0, 1]$: *(2 points)*

$$\sum_{n\in\mathbb{Z}} \Phi(x + 2n - y, t) - \sum_{n\in\mathbb{Z}} \Phi(x + 2n + y, t).$$

(e) Using Poisson's summation formula

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx}$$

and part (a), show the relation

$$H_{[0,1]}(x,y,t) = \sum_{n\in\mathbb{Z}} \Phi(x+2n-y,t) - \sum_{n\in\mathbb{Z}} \Phi(x+2n+y,t),$$

where the left hand side the heat kernel in terms of theta functions as given in the lecture script. Thus the method of images gives the same heat kernel as the Fourier series method (of course, the heat kernel is unique). *(2 bonus points)*