Introduction to Partial Differential Equations
Exercise sheet 8

Ross Ogilvie

23th October, 2023

23. Back in the saddle

Suppose that $u \in C^2(\mathbb{R}^2)$ is a harmonic function with a critical point at x_0. Assume that the Hessian of u has non-zero determinant. Show that x_0 is a saddle point. Explain the connection to the maximum principle. *(3 points)*

24. Subharmonic Functions

Let $\Omega \subset \mathbb{R}^n$ be an open and connected region. A continuous function $v : \overline{\Omega} \to \mathbb{R}$ is called *subharmonic* if for all $x \in \Omega$ and $r > 0$ with $B(x,r) \subset \Omega$ it lies below its spherical mean: $v(x) \leq \mathcal{S}[v](x,r)$.

(a) Prove that every subharmonic function obeys the *maximum principle*: If the maximum of v can be found inside Ω then v is constant. *(2 points)*

(b) Suppose that v is twice continuous differentiable. Show that v is subharmonic if and only if $-\Delta v \leq 0$ in Ω. *(3 points)*

(c) Let $u : \overline{\Omega} \to \mathbb{R}$ be a harmonic function. Show that $\|\nabla u\|^2$ is subharmonic. *(2 points)*

(d) Show that $f \circ u$ is subharmonic for any smooth convex function $f : \mathbb{R} \to \mathbb{R}$. *(2 points)*

(e) Let v_1, v_2 be two subharmonic functions. Show that $v = \max(v_1, v_2)$ is subharmonic. *(1 point)*

25. Never judge a book by its cover

Let $\Omega \subset \mathbb{R}^n$ be an open, connected, and bounded subset, and let $f : \Omega \to \mathbb{R}$ and $g_1, g_2 : \partial\Omega \to \mathbb{R}$ be continuous functions. Consider then the two Dirichlet problems

$$-\Delta u = f \text{ on } \Omega, \qquad u|_{\partial\Omega} = g_k,$$

for $k = 1, 2$. Let u_1, u_2 be respective solutions such that they are twice continuously differentiable on Ω and continuous on $\overline{\Omega}$. Show that if $g_1 \leq g_2$ on $\partial\Omega$ then $u_1 \leq u_2$ on Ω. *(4 points)*

26. To be or not to be

Consider the Dirichlet problem for the Laplace equation $\Delta u = 0$ on Ω with $u = g$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^n$ is an open and bounded subset and g is a continuous function. We know from the weak maximum principle that there is at most one solution. In this question we see that for some domains, existence is not guaranteed.

Consider $\Omega = B(0,1)\backslash\{0\}$, so that the boundary $\partial\Omega = \partial B(0,1) \cup \{0\}$ consists of two components. We write $g(x) = g_1(x)$ for $x \in \partial B(0,1)$ and $g(0) = g_2$. Show that there does not exist a solution for $g_1(x) = 0$ and $g_2 = 1$.

Hint. Use Lemma 3.23, even if we haven't reached it in lectures yet. *(3 points)*