Introduction to Partial Differential Equations
Exercise sheet 7

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19. Twirling towards freedom

Let $u \in C^2(\mathbb{R}^n)$ be a harmonic function. Show that the following functions are also harmonic.

(a) $v(x) = u(x+b)$ for $b \in \mathbb{R}^n$.

(b) $v(x) = u(ax)$ for $a \in \mathbb{R}$.

(c) $v(x) = u(Rx)$ for $R(x_1, \dots, x_n) = (-x_1, x_2, \dots, x_n)$ the reflection operator.

(d) $v(x) = u(Ax)$ for any orthogonal matrix $A \in O(\mathbb{R}^n)$.

Together these show that the Laplacian is invariant under *similarities* (Euclidean motions, reflection and rescaling). *(6 points)*

20. Harmonic Polynomials in Two Variables

(a) Let $u \in C^\infty(\mathbb{R}^n)$ be a smooth harmonic function. Prove that any derivative of u is also harmonic. *(1 point)*

(b) Choose any positive degree n. Consider the complex valued function $f_n : \mathbb{R}^2 \to \mathbb{C}$ given by $f_n(x,y) = (x + \iota y)^n$ and let $u_n(x,y)$ and $v_n(x,y)$ be its real and imaginary parts respectively. Show that u_n and v_n are harmonic. *(2 points)*

(c) A *homogeneous polynomial* of degree n in two variables is a polynomial of the form $p = \sum a_k x^k y^{n-k}$. Show that $\partial_x p$ and $\partial_y p$ are homogeneous of degree $n-1$. *(1 point)*

(d) Show that such a homogeneous polynomial of degree n is harmonic if and only if it is a linear combination of u_n and v_n. *(3 bonus points)*

21. Liouville's Theorem

Liouville's theorem (3.9 in the script) says that if u is bounded and harmonic, then u is constant. In this question we give a geometric proof in $\mathbb{R}^2$ using globular means defined when $\overline{B(x,r)} \subset \Omega$ through

$$\mathcal{M}[v](x,r) = \frac{1}{\omega_n r^n} \int_{B(x,r)} v(y)\, \mathrm{d}y.$$

Recall from Theorem 3.5 that if u is harmonic then it obeys $u(x) = \mathcal{M}[u](x,r)$. This beautiful proof comes from the following article Nelson, 1961.

(a) Consider two points a, b in the plane which are distance $2d$ apart. Now consider two balls, both with radius $r > d$, centred on the two points respectively. Show that the area of the intersection is *(2 bonus points)*

$$\operatorname{area} B(a,r) \cap B(b,r) = 2r^2 \operatorname{acos}(dr^{-1}) - 2d\sqrt{r^2 - d^2}$$

(b) Suppose that v is a bounded function on the plane: $-C \le v(x) \le C$ for all x and some constant C. Show that *(2 points)*

$$\left|\mathcal{M}[v](a,r) - \mathcal{M}[v](b,r)\right| \le \frac{2C}{\omega_2}\left(\pi - 2\operatorname{acos}(dr^{-1}) - \frac{2d}{r}\sqrt{1 - d^2 r^{-2}}\right)$$

(c) Let $u \in C^2(\mathbb{R}^2)$ be a bounded harmonic function. Complete the proof that u is constant. *(2 points)*

22. Weak Tea

In this question we try to generalise the idea of spherical means to distributions in the way suggested at in lectures. Let $\lambda_\varepsilon : \mathbb{R} \to \mathbb{R}$ be the standard mollifier and define a 'sphere mollifier'

$$\Lambda_{x,r,\varepsilon}(y) = \frac{\lambda_\varepsilon(|y-x|-r)}{n\omega_n\,|y-x|^{n-1}}.$$

(a) Describe the support of $\Lambda_{x,r,\varepsilon}$. *(1 point)*

(b) We may try to define the spherical mean of a distribution F as $\lim_{\varepsilon\to 0} F(\Lambda_{x,r,\varepsilon})$. However this does not always exist. Let G be the distribution in Exercise 17(d), submanifold integration on the unit circle. Show that $G(\Lambda_{0,1,\varepsilon}) = \lambda_\varepsilon(1)$ and therefore the limit does not exist. *(2 points)*

(c) Show that the limit $\lim_{\varepsilon\to 0} F(\Lambda_{x,r,\varepsilon})$ does exist for all harmonic distributions F. *(2 points)*