Introduction to Partial Differential Equations
Exercise sheet 5

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12. Is this an applied math course?

In economics, the Black-Scholes equation is a PDE that describes the price V of a (European-style) option under some assumptions about the risk and expected return, as a function of time t and current stock price S. The equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S},$$

where r and σ are constants representing the interest rate and the stock volatility respectively. Describe the order of this equation, and whether it is elliptic, parabolic, and/or hyperbolic.

(3 points)

13. Around and around

Consider the unit circle $C = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$. In this question we will evaluate the integral

$$\int_C x \, d\sigma \qquad (*)$$

in two different ways, demonstrating that it does not depend on the choice of parametrisation.

(a) Consider a regular parameterisation Φ of a subset A and a continuous function f on A as in Definition 2.2. Why (or under what conditions) is the integral unchanged by removing a point from A: *(1 bonus point)*

$$\int_A f \, d\sigma = \int_{A\setminus\{a\}} f \, d\sigma.$$

Therefore in order to compute the submanifold integration $(*)$ it is enough to use parameterisation that cover all but finitely many points of C.

(b) Consider the regular parametrisation $\Phi : (0, 2\pi) \to C$ given by $t \mapsto (\cos t, \sin t)$. Compute the integral $(*)$ using this parametrisation. *(2 points)*

(c) Consider upper and lower halves of the circle: $U_1 = \{(x, y) \in C \mid y > 0\}$ and $U_2 = \{(x, y) \in C \mid y < 0\}$. There are obvious parametrisations $\Phi_i : (-1, 1) \to U_i$ given by $\Phi_1(x) = (x, +\sqrt{1-x^2})$ and $\Phi_2(x) = (x, -\sqrt{1-x^2})$. Compute $(*)$ using these parametrisations.

(2 points)

(d) Compute this integral using the divergence theorem. *(2 points)*

14. The Black Spot

Consider the plane $\mathbb{R}^2$, a disc $B_r = \{x^2 + y^2 \leq r^2\}$ and the function $g(x, y) = \ln(x^2 + y^2)$.

(a) Show that the value of the integral

$$\int_{\partial B_r} \nabla g \cdot N \, d\sigma$$

does not depend on the radius r, where N is the outward pointing normal. *(2 points)*

(b) Can you explain this fact using the divergence theorem? Be careful to avoid the singularity of g at $(0,0)$. *(3 points)*

(c) Generalise the argument of (b). Prove for any compact region $\Omega \subset \mathbb{R}^2$ whose boundary is a manifold, that

$$\int_{\partial\Omega} \nabla g \cdot N \, d\sigma = \begin{cases} 4\pi & \text{if } (0,0) \text{ lies in the interior of } \Omega \\ 0 & \text{if } (0,0) \text{ lies in the exterior of } \Omega \end{cases}$$

(2 points)

15. The Music of the Spheres

In this exercise we show the connection between integration over a ball and integration on spheres in $\mathbb{R}^{n+1}$. More precisely

$$\int_{B(0,R)} f(x)\, dx = \int_0^R \left(\int_{\partial B(0,r)} f(x)\, d\sigma(x) \right) dr$$

You may answer this question in full generality for $\mathbb{R}^{n+1}$ or just for $\mathbb{R}^2$, your choice.

(a) Suppose that you have an $(n+1) \times n$ matrix A and a unit vector $b \in \mathbb{R}^{n+1}$ such that b is perpendicular to every column of A. That is $b^T A = 0$. Let $\tilde{A} = (b \mid A)$ be the square matrix with b as the first column. Argue that *(2 bonus points)*

$$(\det \tilde{A})^2 = \det \tilde{A}^T \tilde{A} = \det A^T A.$$

(b) Let $\Phi : U \to \partial B(0,1)$ be a parameterisation of the unit sphere. Observe then that $\Psi : [0,R] \times U \to B(0,R)$ with $\Psi(r,\theta) = r\Phi(\theta)$ is a parameterisation of the ball. Show that the change of variables matrix for Ψ in the integral on the left hand side above has the form of $\tilde{A}$. *(2 points)*

(c) Hence prove the integration formula. *(2 points)*