Introduction to Partial Differential Equations
Exercise sheet 4

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9. Don't cross the streams

Consider the PDE $\partial_x u + 2x\partial_y u = 0$ on the domain $y > 0$ with the boundary condition $u(x,0) = g(x)$.

(a) Show that the boundary hyperplane $\{y = 0\}$ is non-characteristic at $(x, 0)$, except for $x = 0$. *(1 point)*

(b) What condition does the PDE impose on the boundary data g at the point $(0,0)$? *(1 point)*

(c) Determine the characteristic curves of this PDE. *(1 point)*

(d) By considering the y-derivative of u on the boundary hyperplane, show that there is no C^1 solution with the initial data $g(x) = x$. *(2 points)*

10. It's just a jump to the left

In this question we explore some other solutions to the initial value problem from Example 1.10. As we saw, for small t the method of characteristics gives a unique solution

$$u_{t<1}(x,t) = \begin{cases} 1 & \text{for } x < t \\ \frac{x-1}{t-1} & \text{for } t \leq x < 1 \\ 0 & \text{for } 1 \leq x. \end{cases}$$

(a) (Optional) Derive this solution for yourself, for extra practice.

After $t = 1$, the characteristics begin to cross and so the method cannot assign which value u should have at a point (x,t). However, we could still arbitrarily decide to choose a value of one characteristic. Consider therefore

$$v(x,t) = \begin{cases} u_{t<1} & \text{for } t < 1 \\ 1 & \text{for } x < t \\ 0 & \text{for } t \leq x \end{cases}$$

(b) Draw the corresponding characteristics diagram in the (x,t)-plane for this function. *(2 points)*

(c) Describe the graph of discontinuities $y(t)$. Compute the Rankine-Hugonoit condition for v. *(2 points)*

(d) How much mass (i.e. the integral of v over x) is being lost in the system described by v for $t > 1$? *(2 points)*

11. You're not in traffic, you are traffic

In this question we look at an equation similar to Burgers' equation that describes traffic. Let u measure the number of cars in a given distance of road, the car density. We have seen that f should be interpreted as the flux function, the number of things passing a particular point. When there are no other cars around, cars travel at the speed limit s_m. When they are bumper-to-bumper they can't move, call this density u_m.

(a) What properties do you think that f should have? Does $f(u) = s_m u \cdot (1 - u/u_m)$ have these properties? *(2 points)*

(b) Find a function f that meets your conditions, or use the f from the previous part, and write down a PDE to describe the traffic flow. *(1 point)*

(c) Find all solutions that are constant in time. *(2 points)*

(d) Consider the situation of a traffic light at $x = 0$: to the left of the traffic light, the cars are queued up at maximum density. To the right of the traffic light, the road is empty. Now, at time $t = 0$, the traffic light turns green. Give a discontinuous solution that obeys the Rankine-Hugonoit condition, as well as a continuous solution. *(4 points)*