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7. Royale with Cheese

Recall Burgers' equation from Example 1.5 of the lecture script:

$$\dot{u} + u\partial_x u = 0,$$

for $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. In this question we will apply the method of characteristics to solve this equation for the initial condition g(x) = x.

- (a) According to Theorem 1.4, there is a unique C^1 solution to this initial value problem, at least when t is small. For how long does the theorem guarantee that the solution exists uniquely? (1 point)
- (b) Suppose that u is a solution to this equation and suppose that (x(s), t(s)) is a path in the domain of u. What is the s derivative of u along this path? What constraints should we place on the derivatives of x and t? (2 points)
- (c) On an (x, t)-plane, draw the characteristics and describe the behaviour of this solution.

(2 points)

(d) Finally, derive the following solution to the initial value problem: (2 points)

$$u(x,t) = \frac{x}{1+t}$$

- (e) Is this solution well-defined? Check by substitution that actually solves the initial value problem. (2 points)
- (f) Why is the method of characteristics well-suited to solving first order PDEs that are linear in the derivatives? (1 point)

Solution.

- (a) The condition in the theorem depends on the bound $f''(g(x))g'(x) \ge -\alpha$. For this equation, both f''(u) = 1 and g'(x) = 1, so the product is bound below easily by $\alpha = 0$. Hence the theorem says that the unique solution exists for all time.
- (b) By the chain rule,

$$\frac{d}{ds}u(x(s),t(s)) = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial t}\frac{dt}{ds} = \frac{dt}{ds}\dot{u} + \frac{dx}{ds}\partial_x u.$$

If we compare this to the PDE, then we see that we should choose t' = 1, i.e. $t = s + t_0$, and x' = u. Because the initial condition is t = 0, we choose $t_0 = 0$.

(c) With the choice made in the previous part, the function u is constant along the characteristic. Therefore we have $x' = g(x_0) = x_0$, which integrates to $x(s) = x_0s + x_0$. We already computed in the previous part t(s) = s. The characteristics are the rays $x = x_0t + x_0$ for $x_0 \in \mathbb{R}$. The solution takes the value x_0 on the corresponding ray. We can see that the 'mass' (the conserved quantity) is flowing away from the origin. (d) We have see that the characteristic has the equation

$$x = x_0 + x_0 t = x_0 (1+t),$$

and that u is constant along the characteristic. In other words, we know that for all x_0 and t

$$u(x(t), t) = u(x_0(1+t), t) = g(x_0) = x_0.$$

We have now found the solution in implicit form. To make it explicit, rearrange the relation $x = x_0(1 + t)$. Then

$$u(x,t) = x_0 = \frac{x}{1+t}.$$

(e) The solution is clearly well defined for t > 0 and all x. We compute

$$\dot{u} = -\frac{x}{(1+t)^2}, \qquad \partial_x u = \frac{1}{1+t}.$$

Thus we see that the partial differential equation is solved. As too is the initial value u(x, 0) = x = g(x).

- (f) Because such PDEs resemble the chain rule. Hence we can identity the derivatives $x'_i(s)$ of the path with the coefficient functions in PDE and reduce it to a system of ODEs.
- 8. Solving PDEs Solve the initial value problems of the following PDEs using the method of characteristics. You may assume that g is continuously differentiable on the corresponding domain.
 - (a) $x_1\partial_2 u x_2\partial_1 u = u$ on the domain $x_1, x_2 > 0$, with initial condition $u(x_1, 0) = g(x_1)$. (3 points)
 - (b) $x_1\partial_1 u + 2x_2\partial_2 u + \partial_3 u = 3u$ on $x_1, x_2 \in \mathbb{R}, x_3 > 0$, with initial condition $u(x_1, x_2, 0) = g(x_1, x_2)$.

(3 points)

(c) $u\partial_1 u + \partial_2 u = 1$ on the domain $x_1, x_2 > 0$, with initial condition $u(x_1, x_1) = \frac{1}{2}x_1$. (4 points)

Solution. These PDEs are all quasi-linear (discussed in class), so we may use the simplified form of the equations of the method of characteristics.

(a) This PDE is $(-x_2, x_1) \cdot p - z = 0$. The system of ODEs therefore reads in part

$$\dot{x}_1 = -x_2, \quad \dot{x}_2 = x_1,$$

which is the well know system solved by the sinusoidal functions. From the boundary condition $(x_1, 0)$, we should choose $x_{20} = 0$. Therefore $x_2 = x_{10} \sin s$ and $x_1 = x_{10} \cos s$

are the characteristics. Given a point (x_1, x_2) we can determine the parameters of the characteristics as

$$s = \arctan(x_2/x_1), \ x_{10} = \sqrt{x_1^2 + x_2^2} = |x|.$$

The ODE describing the values of u is $\dot{z} = z$, which has the solution $z(s) = e^s z(0)$. From the initial condition

$$z(0) = u(x(0)) = u(x_{10}, 0) = g(x_{10})$$

Putting this together

$$u(x) = e^{\arctan(x_2/x_1)}g(|x|).$$

(b) From $F = (x_1, 2x_2, 1) \cdot p - 3z$ it follows that

$$x(s) = (x_{10}e^s, x_{20}e^{2s}, x_{30} + s) = (x_{10}e^s, x_{20}e^{2s}, s),$$

where we choose our starting points for the characteristics to lie in the case of the initial conditions, which requires us to set $x_{30} = 0$. Already we can determine the appropriate parameter values for any point: $s = x_3$, $x_{10} = x_1 e^{-x_3}$, and $x_{20} = x_2 e^{-2x_3}$. The the equation for the values is $z = z(0)e^{3s}$, so

$$u(x) = e^{3s}u(x_{10}, x_{20}, 0) = e^{3x_3}g(x_1e^{-x_3}, x_2e^{-2x_3}).$$

(c) This PDE, $F = (z, 1) \cdot p - 1$ is a little different to the others, because of the z in the coefficients of p. This creates a linkage in the system of ODEs:

$$\dot{x}_1 = z, \ \dot{x}_2 = 1, \ \dot{z} = 1.$$

Fortunately, we can solve for z first this time quite easily: z(s) = s + z(0). Then $x(s) = (\frac{1}{2}s^2 + sz(0) + x_{10}, s + x_{20})$. Choose $x_{20} = x_{10}$. This choice means that $x_{10} = x_2 - s$. The initial conditions give $z(0) = u(x(0)) = u(x_{10}, x_{10}) = \frac{1}{2}x_{10}$. Together this allows us to solve for s:

$$x_{1} = \frac{1}{2}s^{2} + s\frac{1}{2}(x_{2} - s) + x_{2} - s$$
$$x_{1} - x_{2} = \frac{1}{2}x_{2}s - s$$
$$s = \frac{2x_{1} - 2x_{2}}{x_{2} - 2}.$$

Finally, what we are interested in is the value of the solution u on these curves, and $u(x(s)) = z(s) = s + z(0) = s + \frac{1}{2}(x_2 - s)$, ie

$$u(x) = \frac{1}{2}x_2 + \frac{1}{2}\frac{2x_1 - 2x_2}{x_2 - 2}$$