Introduction to Partial Differential Equations
Exercise sheet 3

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7. Royale with Cheese

Recall Burgers' equation from Example 1.5 of the lecture script:

$$\dot{u} + u\partial_x u = 0,$$

for $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. In this question we will apply the method of characteristics to solve this equation for the initial condition $g(x) = x$.

(a) According to Theorem 1.4, there is a unique C^1 solution to this initial value problem, at least when t is small. For how long does the theorem guarantee that the solution exists uniquely? *(1 point)*

(b) Suppose that u is a solution to this equation and suppose that $(x(s), t(s))$ is a path in the domain of u. What is the s derivative of u along this path? What constraints should we place on the derivatives of x and t? *(2 points)*

(c) On an (x, t)-plane, draw the characteristics and describe the behaviour of this solution. *(2 points)*

(d) Finally, derive the following solution to the initial value problem: *(2 points)*

$$u(x, t) = \frac{x}{1 + t}.$$

(e) Is this solution well-defined? Check by substitution that actually solves the initial value problem. *(2 points)*

(f) Why is the method of characteristics well-suited to solving first order PDEs that are linear in the derivatives? *(1 point)*

8. Solving PDEs Solve the initial value problems of the following PDEs using the method of characteristics. You may assume that g is continuously differentiable on the corresponding domain.

(a) $x_1\partial_2 u - x_2\partial_1 u = u$ on the domain $x_1, x_2 > 0$, with initial condition $u(x_1, 0) = g(x_1)$. *(3 points)*

(b) $x_1\partial_1 u + 2x_2\partial_2 u + \partial_3 u = 3u$ on $x_1, x_2 \in \mathbb{R}$, $x_3 > 0$, with initial condition $u(x_1, x_2, 0) = g(x_1, x_2)$. *(3 points)*

(c) $u\partial_1 u + \partial_2 u = 1$ on the domain $x_1, x_2 > 0$, with initial condition $u(x_1, x_1) = \frac{1}{2}x_1$. *(4 points)*