Introduction to Partial Differential Equations
Exercise sheet 2

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4. Inhomogeneous Transport Equation First order partial differential equations share many things in common with first order ordinary differential equations (ODEs). Consider the linear inhomogeneous ODE

$$\frac{du}{dt} = f(t).$$

(a) Find a solution $u : \mathbb{R} \to \mathbb{R}$ to this equation. *(1 point)*

(b) For any initial value $c \in \mathbb{R}$, show that there is a unique solution with $u(0) = c$. *(2 points)*

We consider now the inhomogeneous transport equation

$$\partial_t u + b \cdot \nabla u = f$$

with initial value given by a function $g(x)$, namely $u(x, 0) = g(x)$. It had an explicit solution

$$u(x,t) = g(x - tb) + \int_0^t f(x + (s - t)b, s)\, ds.$$

(c) Show that the integral term itself solves the inhomogeneous transport equation. What initial value problem does it solve? *(3 points)*

(d) Prove that the solution to the initial value problem is unique. (You may assume that the solution to the homogeneous version is unique.) *(2 points)*

5. Method of characteristics for an Inhomogeneous PDE Use the method of characteristics to solve the following *inhomogeneous* PDE

$$x\partial_x u + y\partial_y u = 2u$$

on the domain $x > 0, y \in \mathbb{R}$, with initial condition $u(1, y) = y$. Note, the function u will *not* be constant along the characteristic, but its value along the characteristic will be determined by its initial value. *(6 points)*

6. Duhamel's Principle

Duhamel's principle is a method to find a solution to an inhomogeneous PDE if one can solve the homogeneous PDE for any initial condition. In this exercise we give the general idea and show how it applies to the transport equation. Consider an inhomogeneous PDE on $\mathbb{R}^n \times \mathbb{R}$ of the following form

$$\partial_t u - Lu = f(x,t), \quad u(x, 0) = 0,$$

where L is a linear differential operator on $\mathbb{R}^n$ with constant coefficients. The idea is to instead consider the following family of homogeneous equations

$$\partial_t u_s - Lu_s = 0, \quad u_s(x, s) = f(x, s).$$

Suppose that we can find such solutions u_s. Prove that

$$u(x,t) = \int_0^t u_s(x,t)\, ds$$

is a solution to the inhomogeneous problem. (Do not worry about convergence problems.)
Use this method to solve the inhomogeneous transport. *(2 + 4 points)*