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Introduction to Partial Differential Equations
Exercise sheet 14

6th December, 2022

48. Method of Descent

In this exercise we will apply the method of descent to solve the wave equation on $\mathbb{R}^2$ for a particular set of initial conditions. The idea is to help you understand the key ideas and notation of the method. It is a combination of results from Sections 5.1–4.

Consider the wave equation on $\mathbb{R}^2$ with initial conditions

$$\begin{aligned} &\partial_t^2 u - \Delta u = 0 \text{ on } (x,t) \in \mathbb{R}^2 \times (0,\infty), \\ &u(x,0) = g(x) = \chi_{[0,\infty)}(x_1), \qquad \partial_t u(x,0) = h(x) = 0. \end{aligned}$$

(a) In Section 5.4 we define $\bar{u} : \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}$ associated to the function u. Explain why $\bar{g}(x_1, x_2, x_3) = \chi_{[0,\infty)}(x_1)$ and $\bar{h} = 0$ (or give the definition of bar). Why does $\bar{u}$ solve the wave equation on $\mathbb{R}^3$? (note, the Laplacians are different in different dimensions). *(2 points)*

(b) Conversely, prove that a solution $\bar{u}$ to the 3-dimensional wave equation that does not depend on x_3 gives a solution to the 2-dimensional wave equation. *(2 points)*

(c) By (a) and (b), we now must solve a wave equation on $\mathbb{R}^3$. The key to solving the 3-dimensional wave equation is to consider the (spatial-)spherical means

$$U(x,t,r) = \frac{1}{4\pi r^2} \int_{\partial B(x,r)} \bar{u}(z,t)\, d\sigma(z),$$

and likewise let G and H be the spherical means of $\bar{g}$ and $\bar{h}$ respectively. Show that

$$G(x,r) = \begin{cases} 0 & \text{for } x_1 \leq -r \\ \frac{1}{2}\frac{x_1 + r}{r} & \text{for } |x_1| \leq r \\ 1 & \text{for } r \leq x_1 \end{cases} \qquad \text{and } H(x,r) = 0.$$

You may use the following geometric fact: for $-R < a < b < R$, the surface area of the part of the sphere $\partial B(0,R)$ with $a < x_1 < b$ is $2\pi R(b-a)$. *(4 points)*

(d) We know by Lemma 5.2 that U obeys the Euler-Poisson-Darboux equation. Let $\tilde{U}(x,t,r) := rU(x,t,r)$. Show that $\tilde{U}$ obeys the following PDE

$$\begin{aligned} &\partial_t^2 \tilde{U} - \partial_r^2 \tilde{U} = 0 \text{ on } (t,r) \in [0,\infty) \times [0,\infty), \\ &\tilde{U}(x,0,r) = rG(x,r), \qquad \partial_t \tilde{U}(x,0,r) = rH(x,r). \end{aligned}$$

Note that x plays no role in this PDE, so we can think of it as a family of PDEs parametrised by x. *(2 points)*

(e) Thus we see that $\tilde{U}$ obeys the 1-dimensional wave equation on the *half-line* $r \in [0,\infty)$. This is solved by a trick using reflection, and the formula is at the end of Section 5.1. We only need the solution for small r, so it is enough to consider the case $0 \leq r \leq t$. In this case, show

$$\tilde{U}(x,t,r) = \begin{cases} 0 & \text{for } x_1 \leq -(t+r) \\ \frac{1}{4}(x_1+t+r) & \text{for } -(t+r) \leq x_1 \leq -(t-r) \\ \frac{1}{2}r & \text{for } |x_1| \leq t-r \\ \frac{1}{4}(x_1-t+3r) & \text{for } t-r \leq x_1 \leq t+r \\ r & \text{for } x_1 \geq t+r. \end{cases}$$

(4 points)

(f) Recover $\bar{u}$ from $\tilde{U}$ using a certain property of spherical means. *(3 points)*

Observe that $\bar{u}$ does not depend on x_3. So by part (b) we have a solution to the 2-dimensional wave equation:

$$u(x_1,x_2,t) = \begin{cases} 0 & \text{for } x_1 < -t \\ 0.25 & \text{for } x_1 = -t \\ 0.5 & \text{for } -t < x_1 < t \\ 0.75 & \text{for } x_1 = t \\ 1 & \text{for } x_1 > t. \end{cases}$$

This solution has jump discontinuities, but this is unsurprising since the initial conditions also had them.

49. Plane Waves.

Suppose that $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a solution to the following modified wave equation:

$$\frac{\partial^2 u}{\partial t^2} - \sum_{j=1}^{n} c_j^2 \frac{\partial^2 u}{\partial x_j^2} = 0\ , \tag{$*$}$$

where $c_1, \ldots, c_n > 0$ are constants.

(a) Let $\alpha \in \mathbb{R}^n$ be a unit vector $\|\alpha\| = 1$, $\mu \in \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ a twice continuously differentiable function. Show that

$$u(x,t) := F(\alpha \cdot x - \mu t)$$

is a solution of $(*)$ exactly when

$$\mu^2 = \sum_{j=1}^{n} \alpha_j^2 c_j^2$$

or F is linear. Solutions of $(*)$ with this form are called *plane waves.* *(2 points)*

(b) For the solutions in (a), examine whether the following property holds for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$:

$$u(x,t) = u(x - \mu t \alpha, 0).$$

Interpret this equation in terms of direction and speed. *(3 points)*

50. Electromagnetic Waves.

In physics, electrical and magnetic fields are modelled as time-dependent vector fields, which mathematically are smooth functions $E, B : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$. Through a series of experiments in the 18th and 19th centuries, the existence and properties of these fields were discovered. Importantly, it was discovered that the two phenomena were connected (both magnets and static electricity had been known since antiquity). In 1861 James Clerk Maxwell published a series of papers summarising electromagnetic theory, including a collection of 20 differential equations. Over time these were further reduced to the following four (by Heaviside 1884 using vector notation), called *Maxwell's Equations*:

$$\nabla \cdot E = \frac{1}{\varepsilon_0}\rho \qquad \nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \cdot B = 0 \qquad \nabla \times B = \mu_0 J + \varepsilon_0 \mu_0 \frac{\partial E}{\partial t}.$$

As is usual, the ∇ operator acts on the spatial coordinates x, and the $\times$ denotes the cross product of $\mathbb{R}^3$. The constants ε_0, the electrical permittivity, and μ_0, the magnetic permeability, are approximately $\varepsilon_0 \approx 8,854 \cdot 10^{-12} \frac{\mathrm{A \cdot s}}{\mathrm{V \cdot m}}$ and $\mu_0 \approx 1,257 \cdot 10^{-6} \frac{\mathrm{V \cdot s}}{\mathrm{A \cdot m}}$ (V=Volt, s=Seconds, A=Ampere and m=Metre) in a vacuum. Electrical charges are included via the charge density ρ and electric currents are the movements of charges, $J := v\rho$ for a velocity field v.

The two equations with divergence were formulated by Gauss, the curl of the electric field is due to Faraday, and the curl of the magnetic field is due to Ampère. The last term in Ampère's law that has the time-derivative of the electrical field was an addition of Maxwell. With this correction, he was able to derive the equations for electromagnetic waves, as you will now do.

(a) Let E und B be solutions to Maxwell's equations in the absence of electric charges, $\rho = 0, J = 0$. Show that they each satisfy a modified wave equation (Question 41). You may use without proof the identity $\nabla \times (\nabla \times f) = \nabla(\nabla \cdot f) - \triangle f$ for smooth functions $f : \mathbb{R}^3 \to \mathbb{R}^3$. *(3 points)*

(b) Predict the speed of these waves. *((2 Bonus Points))*

(c) Argue that Ampère's law in its original form $\nabla \times B = \mu_0 J$ violates the conservation of charge ρ under some conditions. Refer to Exercise Sheet 5 for the definition of a conservation law. Thereby derive Maxwell's additional term. *((3 Bonus Points))*