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Introduction to Partial Differential Equations
Exercise sheet 13

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45. 1D Waves.

(a) Show that the a smooth function $u = u(\zeta, \eta) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a solution to $\partial_{\zeta\eta} u = 0$ exactly when it is of the form $u(\zeta, \eta) = F(\zeta) + G(\eta)$, for smooth functions $F, G : \mathbb{R} \to \mathbb{R}$. *(2 points)*

(b) Under the parameterisation $\zeta = x + t, \eta = x - t$, show that u obeys the one dimensional wave equation $(\partial_{tt} - \partial_{xx})u = 0$ exactly when it solves the PDE in (a). *(3 points)*

(c) From parts (a) and (b), derive D'Alembert's formula. *(2 points)*

Solution.

(a) Suppose u is a solution. Integrating once, we see that $\partial_\eta u = g(\eta)$, because for each value of η, $\partial_\eta u$ must be constant in ζ. Integrating again gives $u = F(\zeta) + \int g(\eta)\, d\eta =: F(\zeta) + G(\eta)$. The converse is immediate.

(b) Using the chain rule we compute the operator under the change of variables. Note $x = \frac{1}{2}(\zeta + \eta)$ and $y = \frac{1}{2}(\zeta - \eta)$.

$$\frac{\partial}{\partial \zeta} = \frac{\partial x}{\partial \zeta}\frac{\partial}{\partial x} + \frac{\partial t}{\partial \zeta}\frac{\partial}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial t}$$
$$\frac{\partial}{\partial \eta} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2}\frac{\partial}{\partial t}$$
$$\partial_{\zeta\eta} = \frac{1}{4}(\partial_x - \partial t)(\partial_x + \partial t) = \frac{1}{4}(\partial_{xx} - \partial xt + \partial_{tx} - \partial tt) = \frac{1}{4}(\partial_{xx} - \partial tt).$$

We see then that the operator $\partial_{tt} - \partial_{xx}$ is just a rescaling of $\partial_{\zeta\eta}$.

(c) We are asked to solve the problem of Theorem 5.1:

$$\begin{cases} \partial_{tt} u - \partial_{xx} u = 0 \\ u(x, 0) = g(x) \\ \partial_t u(x, 0) = h(x), \end{cases}$$

where g is twice continuously differentiable and f is only once continuously differentiable. From (a), we know that the equation is simpler in the (ζ, η) coordinates, where $u(\zeta, \eta) = F(\zeta) + G(\eta)$ solves the wave equation. The functions F and G are only defined up to a constant between them (ie $u = (F - C) + (G + C)$ also), so without loss of generality choose $G(0) = 0$.

When $t = 0$ that corresponds to $x = \zeta = \eta$. So the initial conditions say $F(\zeta) + G(\zeta) = g(\zeta)$ and $\partial_t u|_{t=0} = (\partial_\zeta - \partial_\eta)u|_{\zeta=\eta} = F'(\zeta) - G'(\zeta) = h(\zeta)$. Integrating the latter gives

$$\int_0^\zeta h(y)\, dy = \int_0^\zeta F'(y) - G'(y)\, dy = F(\zeta) - G(\zeta) - (F(0) - G(0)) = F(\zeta) - G(\zeta) - g(0).$$

Now we have two linear equations for F and G, so solving gives

$$F(\zeta) = \frac{1}{2}\left[g(\zeta) + g(0) + \int_0^\zeta h(y)\, dy\right], \qquad G(\zeta) = \frac{1}{2}\left[g(\zeta) - g(0) - \int_0^\zeta h(y)\, dy\right].$$

Changing the variable in G back to η and summing gives:

$$\begin{aligned} u &= F(\zeta) + G(\eta) \\ &= \frac{1}{2}\left[g(\zeta) + \int_0^\zeta h(y)\,dy\right] + \frac{1}{2}\left[g(\eta) - \int_0^\eta h(y)\,dy\right] \\ &= \frac{1}{2}\left[g(\zeta) + g(\eta)\right] + \frac{1}{2}\int_\eta^\zeta h(y)\,dy. \end{aligned}$$

Finally, changing back to (x, t) coordinates gives the desired formula.

46. Faster!

How should you modify D'Alembert's formula for this situation?

$$\begin{cases} \partial_{tt}u - a^2\partial_{xx}u = 0 \\ u(x,0) = g(x) \\ \partial_t u(x,0) = h(x), \end{cases}$$

Solve this for the initial data $a = 2$, $g(x) = \sin(x)$ and $h(x) = 1$. *(5 points)*

Solution. One can rescale one of the coordinates to compensate for the factor of a^2. Namely, let $\tau = at$. Because $t = 0$ when $\tau = 0$, the first initial condition is unchanged. The second initial condition however reads $a\partial_\tau u(x, 0) = h(x)$. Using the formula for the solution to this new initial value problem for the waver equation, but then further making the substitution $\tau = at$, gives

$$u(x,t) = \frac{1}{2}\left[g(x+at) + g(x-at)\right] + \frac{1}{2}\int_{x-at}^{x+at} \frac{1}{a} h(y)\,dy.$$

With the given initial data

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[\sin(x+2t) + \sin(x-2t)\right] + \frac{1}{4}\int_{x-2t}^{x+2t} 1\,dy \\ &= \frac{1}{2}\left[\sin(x+2t) + \sin(x-2t)\right] + t. \end{aligned}$$

47. The energy of solutions of the wave equation.

Let $g, h : \mathbb{R} \to \mathbb{R}$ be smooth functions and $u : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a solution to the initial value problem

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad u(\cdot, 0) = g, \quad \text{and} \quad \frac{\partial u}{\partial t}(\cdot, 0) = h.$$

We define $k(t)$ and $p(t)$, called respectively the *kinetic energy* and the *potential energy* of the solution at time t, by the formulae

$$k(t) := \tfrac{1}{2}\int_{-\infty}^{\infty}\left(\frac{\partial u}{\partial t}(x,t)\right)^2 dx \quad \text{and} \quad p(t) := \tfrac{1}{2}\int_{-\infty}^{\infty}\left(\frac{\partial u}{\partial x}(x,t)\right)^2 dx.$$

(a) Consider the standing sinusoidal wave $u(x,t) = \sin t \sin x$. Compute its kinetic and potential energy on the interval $[0, \pi]$. *(2 points)*

(b) *(Conversation of energy.)* Suppose the total energy $E(t) = k(t) + p(t)$ is finite. Show that it is constant over time. *(3 points)*

(c) Prove that solutions with finite energy to this initial value problem are unique (if they exist). *(3 points)*

We now suppose that the initial conditions g, h have compact support.

(d) Show that $u(\cdot, t)$ has compact support for every $t > 0$. Hence the total energy is finite. *(2 points)*

(e) Show that for sufficiently large t, the functions $k(t)$ and $p(t)$ are each constant. *(3 points)*

Note: A student pointed out that my intended solution doesn't work. Total integral finite is not enough to force the limit of the derivatives to be zero, which is needed to show that the energy is constant. Therefore assume that u has compact support for all the questions. **Solution.**

(a) $\partial_t u = \cos t \sin x$. Therefore

$$k(t) = \frac{1}{2}\int_0^{\pi} \cos^2 t \sin x^2 \; dx = \frac{\pi}{4}\cos^2 t.$$

In a similar way $p(t) = \frac{\pi}{4}\sin^2 t$. Here we already see that the total energy is constant, it just shifts back and forth between kinetic and potential.

(b) Because the integrals are finite, they are dominated on bounded time intervals, and so we may differentiate under the integral sign. This gives

$$\begin{aligned}
\partial_t E &= \int_{-\infty}^{\infty} \partial_t u \, \partial_{tt} u + \partial_x u \, \partial_{xt} u \; dx \\
&= \int_{-\infty}^{\infty} \partial_t u \, \partial_{xx} u + \partial_x u \, \partial_{xt} u \; dx \\
&= \int_{-\infty}^{\infty} \partial_x \Big(\partial_t u \, \partial_x u\Big) \; dx \\
&= \partial_t u \, \partial_x u \Big|_{-\infty}^{\infty} = 0.
\end{aligned}$$

(c) Suppose that there were two such solutions with finite energy. Then the difference would also obey the wave equation with initial conditions $g = h = 0$. Hence the total energy is initially zero. From part (a), the total energy is constant, so we conclude that both first partial derivatives are zero. Hence the difference of the two solutions is everywhere constant. Evaluating at the initial conditions shows the solutions are equal.

(d) D'Alembert's formula says

$$u(x,t) = \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy.$$

The terms with $g(x+t)$ and $g(x-t)$ clearly have compact support because they are just shifts of g, which itself has compact support. Let the support of $h(y)$ be contained in $[-R, R]$. Then for $x > R+t$ or $x < -R-t$ the entire interval $[x-t, x+t]$ lies outside the support of h. Hence the integral term is zero for large $|x|$.

The derivatives of a function with compact support also have compact support, hence the comment that the energies are finite.

(e) Suppose that both g and h are supported on $[-R, R]$. We calculate using D'Alembert's formula:

$$\begin{aligned} 8k(t) &= \int_{-\infty}^{\infty} \left[g'(x+t) - g'(x-t) + h(x+t) + h(x-t)\right]^2\, dx \\ &= \int_{-\infty}^{\infty} \left[(g'(x+t) + h(x+t)) + (-g'(x-t) + h(x-t))\right]^2\, dx \\ &= \int_{-\infty}^{\infty} (g'(x+t) + h(x+t))^2 + 0 + (-g'(x-t) + h(x-t))^2\, dx, \end{aligned}$$

for $t > R$ because then the supports of $g'(x+t) + h(x+t)$ and $(-g'(x-t) + h(x-t)$ cannot overlap. Continuing:

$$\begin{aligned} 8k(t) &= \int_{-\infty}^{\infty} (g'(x+t) + h(x+t))^2\, dx + \int_{-\infty}^{\infty} (-g'(x-t) + h(x-t))^2\, dx \\ &= \int_{-\infty}^{\infty} (g'(y) + h(y))^2\, dy + \int_{-\infty}^{\infty} (-g'(z) + h(z))^2\, dz \end{aligned}$$

shows that the kinetic energy is independent of t.

The proof for the potential energy is similar.