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**Introduction to Partial Differential Equations**
**Exercise sheet 13**

29th November, 2022

**45. 1D Waves.**

**(a)** Show that the a smooth function $u = u(\zeta, \eta) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a solution to $\partial_{\zeta\eta} u = 0$ exactly when it is of the form $u(\zeta, \eta) = F(\zeta) + G(\eta)$, for smooth functions $F, G : \mathbb{R} \to \mathbb{R}$. *(2 points)*

**(b)** Under the parameterisation $\zeta = x + t, \eta = x - t$, show that $u$ obeys the one dimensional wave equation $(\partial_{tt} - \partial_{xx})u = 0$ exactly when it solves the PDE in (a). *(3 points)*

**(c)** From parts (a) and (b), derive D'Alembert's formula. *(2 points)*

**46. Faster!**

How should you modify D'Alembert's formula for this situation?

$$\begin{cases} \partial_{tt} u - a^2 \partial_{xx} u = 0 \\ u(x, 0) = g(x) \\ \partial_t u(x, 0) = h(x), \end{cases}$$

Solve this for the initial data $a = 2$, $g(x) = \sin(x)$ and $h(x) = 1$. *(5 points)*

**47. The energy of solutions of the wave equation.**

Let $g, h : \mathbb{R} \to \mathbb{R}$ be smooth functions and $u : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a solution to the initial value problem

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad u(\cdot, 0) = g, \quad \text{and} \quad \frac{\partial u}{\partial t}(\cdot, 0) = h.$$

We define $k(t)$ and $p(t)$, called respectively the *kinetic energy* and the *potential energy* of the solution at time $t$, by the formulae

$$k(t) := \tfrac{1}{2} \int_{-\infty}^{\infty} \left( \frac{\partial u}{\partial t}(x, t) \right)^2 dx \quad \text{and} \quad p(t) := \tfrac{1}{2} \int_{-\infty}^{\infty} \left( \frac{\partial u}{\partial x}(x, t) \right)^2 dx.$$

**(a)** Consider the standing sinusoidal wave $u(x, t) = \sin t \sin x$. Compute its kinetic and potential energy on the interval $[0, \pi]$. *(2 points)*

**(b)** *(Conversation of energy.)* Suppose the total energy $E(t) = k(t) + p(t)$ is finite. Show that it is constant over time. *(3 points)*

**(c)** Prove that solutions with finite energy to this initial value problem are unique (if they exist). *(3 points)*

We now suppose that the initial conditions $g, h$ have compact support.

**(c)** Show that $u(\cdot, t)$ has compact support for every $t > 0$. Hence the total energy is finite. *(2 points)*

**(d)** Show that for sufficiently large $t$, the functions $k(t)$ and $p(t)$ are each constant. *(3 points)*