

42. The heat kernel on $[0, 1]$.

(Exercise 4.23 from the lecture script)

- (a) Show the final step in the calculation of the heat kernel $H_{[0,1]}$: (2 points)

$$\sum_{k=1}^{\infty} e^{-\pi^2 k^2 t} (\cos(k\pi(x-y)) - \cos(k\pi(x+y))) = \frac{1}{2} \Theta\left(\frac{x-y}{2}, \pi it\right) - \frac{1}{2} \Theta\left(\frac{x+y}{2}, \pi it\right)$$

- (b) Let \mathcal{A} be the space of all continuous functions on \mathbb{R} with the following properties:

$$f(n+x) = \begin{cases} f(x) & \text{for even } n \in 2\mathbb{Z} \text{ and } x \in \mathbb{R} \\ -f(1-x) & \text{for odd } n \in 2\mathbb{Z} + 1 \text{ and } x \in \mathbb{R}. \end{cases}$$

Show that the functions in \mathcal{A} vanish at \mathbb{Z} and that \mathcal{A} contains all continuous odd and periodic functions with period 2. (1 point)

- (c) Show that for any Schwartz function f on \mathbb{R} the following series converges to a smooth functions \tilde{f} in \mathcal{A} : (2 points)

$$\tilde{f}(x) = \sum_{n \in \mathbb{Z}} f(2n+x) - \sum_{n \in \mathbb{Z}} f(2n-x).$$

- (d) Conclude that the following sum is a heat kernel of $[0, 1]$: (3 points)

$$\sum_{n \in \mathbb{Z}} \Phi(x+2n-y, t) - \sum_{n \in \mathbb{Z}} \Phi(x+2n+y, t).$$

- (e) Using Poisson's summation formula

$$\sum_{n \in \mathbb{Z}} f(x+n) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n x}$$

and part (a), show the relation

$$H_{[0,1]}(x, y, t) = \sum_{n \in \mathbb{Z}} \Phi(x+2n-y, t) - \sum_{n \in \mathbb{Z}} \Phi(x+2n+y, t),$$

where the left hand side the heat kernel in terms of theta functions as given in the lecture script. Thus the method of images gives the same heat kernel as the Fourier series method.

(2 bonus points)

Solution.

- (a) We begin with a theta function:

$$\begin{aligned} \Theta\left(\frac{z}{2}, \pi it\right) &= \sum_{k \in \mathbb{Z}} \exp(\pi i k z + (\pi i)^2 t k^2) \\ &= \sum_{k \in \mathbb{Z}} e^{-\pi^2 t k^2} \exp(\pi i k z) \\ &= 1 + \sum_{k=1}^{\infty} e^{-\pi^2 t k^2} [\exp(\pi i k z) + \exp(-\pi i k z)] \\ &= 1 + \sum_{k=1}^{\infty} e^{-\pi^2 t k^2} 2 \cos(\pi k z). \end{aligned}$$

Taking the difference with $z = x - y$ and $z = x + y$ gives the the result.

- (b) The functions of \mathcal{A} are clearly periodic with period 2, since for $n = 2$ we have the relation $f(2 + x) = f(x)$. For $n = 1$ we have that $f(1 + x) = -f(1 - x) = -f(-(1 + x))$, which shows that these functions are odd. Therefore it vanishes at 0 and all even integers. Setting $x = 0$ also gives $f(1 + 0) = -f(1 - 0)$ showing it vanishes at 1, and hence all odd integers also.

Now take any odd function f with period 2. Then clearly we have $f(n + x) = f(x)$ for any even integer n ; this is the definition of ‘period 2’. For an odd integer n :

$$f(n + x) = f(1 + x) = -f(-(1 + x)) = -f(-1 - x) = -f(1 - x).$$

This shows $f \in \mathcal{A}$.

- (c) Just consider one sum. As we discussed in the exercise that asked for the definition of a Schwartz function, we know that $f(x) \leq C(1 + x^2)^{-1}$. This is a sufficient bound to show that the sum exists for all x . Moreover, we have uniform convergence on compact subsets, and so the resulting function is also smooth.

By shifting the summation indices we see that it has period 2 also. It remains to show that it is an odd function

$$\tilde{f}(-x) = \sum_{n \in \mathbb{Z}} f(2n - x) - \sum_{n \in \mathbb{Z}} f(2n + x) = -\tilde{f}(x).$$

- (d) We have to check that it fits Definition 4.14 with $\Omega = (0, 1)$. For positive time the fundamental solution is smooth and Schwartz. Thus the sums are smooth too and defined on all of $x, y \in \mathbb{R}$. Thus there is no problem extending to the boundary. We need to check that it vanishes on $\partial\Omega = \{0, 1\}$. For $y = 0$ this is clear. For $y = 1$, relabel the second index $n = m - 1$ to see the two sums are also equal.

Now we need to verify condition (ii). Consider

$$H_{[0,1]}(x, y, t) - \Phi(x - y, t) = \sum_{n \in \mathbb{Z}, n \neq 0} \Phi(x + 2n - y, t) - \sum_{n \in \mathbb{Z}} \Phi(x + 2n + y, t).$$

We know that $\Phi(z, t)$ extends continuously with value 0 as $t \rightarrow 0$ if $z \neq 0$ (at $(0, 0)$ there is a singularity). If $x \in \Omega = (0, 1)$ and $y \in \bar{\Omega} = [0, 1]$ then $x - y \in (-1, 1)$ and so $x + 2n - y \neq 0$ for $n \neq 0$. Similarly $x + y \in (0, 2)$ so the singularities are avoided in the second sum too.

- (e) We have previously calculated the Fourier transform of $\exp(-A^2 x^2)$, so using $A^{-2} = 4t$, gives

$$\hat{\Phi}(k, t) = \frac{1}{\sqrt{4\pi t}} \sqrt{\pi} 2\sqrt{t} \exp(-\pi^2 k^2 4t) = \exp(-4\pi^2 k^2 t)$$

Now use the hint applied to a sum of fundamental solutions

$$\sum_{n \in \mathbb{Z}} \Phi(z + n, t) = \sum_{n \in \mathbb{Z}} e^{-4\pi^2 n^2 t} e^{2\pi i n z} = \sum_{n \in \mathbb{Z}} e^{i\pi^2 n^2 (4it)} e^{2\pi i n z} = \Theta(z, 4\pi i t).$$

Now we use the theta function version of the heat kernel

$$\begin{aligned} H_{(0,1)}(x, y, t) &= \frac{1}{2}\Theta\left(\frac{x-y}{2}, \pi it\right) - \frac{1}{2}\Theta\left(\frac{x+y}{2}, \pi it\right) \\ &= \sum_{n \in \mathbb{Z}} \frac{1}{2}\Phi\left(\frac{x-y}{2} + n, \frac{t}{4}\right) - \sum_{n \in \mathbb{Z}} \frac{1}{2}\Phi\left(\frac{x+y}{2} + n, \frac{t}{4}\right) \\ &= \sum_{n \in \mathbb{Z}} \Phi(x - y + 2n, t) - \sum_{n \in \mathbb{Z}} \Phi(x + y + 2n, t), \end{aligned}$$

using the rescaling property of the fundamental solution.

43. Scaling the heat kernel.

Find the heat kernel of

(a) $\mathbb{R}/c\mathbb{R}$ with $c > 0$, (2 points)

(b) $[a, b]$ with $-\infty < a < b < \infty$. (2 points)

Solution.

- (a) To begin, think about what functions there are between $\mathbb{R}/c\mathbb{Z}$ and \mathbb{R}/\mathbb{Z} . The obvious one is $x \mapsto x/c$. How does the fundamental solution for $n = 1$ behave under scaling then?

$$\Phi(cx, c^2t) = \frac{1}{\sqrt{4\pi c^2t}} \exp\left(-\frac{(cx)^2}{4c^2t}\right) = \frac{1}{c}\Phi(x, t).$$

So the fundamental solution on $\mathbb{R}/c\mathbb{Z}$ is $\frac{1}{c}\tilde{\Phi}(z/c, t/c^2)$. The heat kernel is just the shifted version of the fundamental solution, as there are no boundary conditions on the circle.

- (b) In a similar way, one transformation from $[a, b]$ to $[0, 1]$ is $\frac{x-a}{b-a}$. We already know how the fundamental solution behaves under scaling, but it does not simplify when shifted. However, remember that the shifting behaviour is built into the idea of the heat kernel/Greens function, namely $G(x, y) = \phi(x - y)$. If we set $H_{[a,b]}(x, y, t) = \frac{1}{b-a}H_{[0,1]}\left(\frac{x-a}{b-a}, \frac{y-a}{b-a}, \frac{t}{(b-a)^2}\right)$ then clearly it vanishes at the boundaries $x = a, b$. Further

$$\begin{aligned} H_{[a,b]}(x, y, t) - \Phi(x - y, t) &= \frac{1}{b-a} \left[H_{[0,1]}\left(\frac{x-a}{b-a}, \frac{y-a}{b-a}, \frac{t}{(b-a)^2}\right) - \Phi\left(\frac{x-y}{b-a}, \frac{t}{(b-a)^2}\right) \right] \\ &= \frac{1}{b-a} \left[H_{[0,1]}\left(\frac{x-a}{b-a}, \frac{y-a}{b-a}, \frac{t}{(b-a)^2}\right) - \Phi\left(\frac{x-a}{b-a} - \frac{y-a}{b-a}, \frac{t}{(b-a)^2}\right) \right] \\ &= \frac{1}{b-a} [H_{[0,1]}(x', y', t') - \Phi(x', y', t')], \end{aligned}$$

so this must solve the homogenous heat equation and vanish initially ($t = 0 \Leftrightarrow t' = 0$). Therefore it is the heat kernel.

44. Some like it hot.

Find the solution $u : (0, \pi) \times \mathbb{R}^+ \rightarrow \mathbb{R}$ of the initial and boundary value problem:

$$\begin{cases} \dot{u} - 7\partial_{xx}u = 0 & \text{for } x \in (0, \pi), t > 0 \\ u(0, t) = u(\pi, t) = 0 & \text{for } t > 0 \\ u(x, 0) = 3 \sin(2x) - 6 \sin(5x) & \text{for } x \in (0, \pi). \end{cases}$$

(6 points)

Solution. Firstly, this isn't quite the heat equation, but we can rescale time to absorb the factor 7, namely $s = 7t$. We have just calculated the heat kernel for an interval, so let's use it! By Theorem 4.16

$$\begin{aligned} u(x, s) &= 0 - 0 + \int_{[0, \pi]} [3 \sin(2y) - 6 \sin(5y)] H_{[0, \pi]}(x, y, s) dy \\ &= 0 - 0 + \int_{[0, \pi]} [3 \sin(2y) - 6 \sin(5y)] \frac{1}{\pi} H_{[0, 1]}\left(\frac{x}{\pi}, \frac{y}{\pi}, \frac{s}{\pi^2}\right) dy \\ &= \int_0^\pi [3 \sin(2y) - 6 \sin(5y)] \frac{1}{\pi} \sum_{k=1}^\infty e^{-k^2 s} 2 \sin(kx) \sin(ky) dy \\ &= \frac{2}{\pi} \sum_{k=1}^\infty e^{-k^2 s} \sin(kx) \left[\int_0^\pi 3 \sin(2y) \sin(ky) dy - \int_0^\pi 6 \sin(5y) \sin(ky) dy \right]. \end{aligned}$$

Know it is relatively easy to see, by applying integration by parts twice, that $\int_0^\pi \sin mz \sin nz$ is zero if $m \neq n$ and is $\pi/2$ if they are equal. Thus all but two terms of the sum are zero.

$$\begin{aligned} u(x, s) &= \frac{2}{\pi} e^{-4s} \sin(2x) 3 \frac{\pi}{2} - \frac{2}{\pi} e^{-25s} \sin(5x) 6 \frac{\pi}{2} \\ &= 3e^{-4s} \sin(2x) - 6e^{-25s} \sin(5x) \\ u(x, t) &= 3e^{-28t} \sin(2x) - 6e^{-175t} \sin(5x) \end{aligned}$$

The temperature falls very quickly, so make sure you have a jacket. We can also check our solution:

$$\begin{aligned} \dot{u}(x, t) &= -28 \cdot 3e^{-28t} \sin(2x) + 175 \cdot 6e^{-175t} \sin(5x) \\ \partial_{xx}u(x, t) &= -4 \cdot 3e^{-28t} \sin(2x) + 25 \cdot 6e^{-175t} \sin(5x). \end{aligned}$$

Because this is just the 1-dimensional heat equation, there are a variety of other effective methods: separation of variables to reduce it to two ODEs, or a Laplace transform to reduce it to an inhomogeneous ODE are two that spring to mind.

Another approach to the above integrals would be to write the heat kernel in the form $\hat{\Phi}(n, t)e^{2\pi inx}$. Then again all but two of the integrals will be zero and we see the solution is the sum of two Fourier transforms of the fundamental solution.

45. Out of the frying pan, into the fire.

Find the solution $u : (0, \pi) \times \mathbb{R}^+ \rightarrow \mathbb{R}$ of the initial and boundary value problem:

$$\begin{cases} u_t - \partial_{xx}u = 0 & \text{for } x \in (0, \pi), t > 0 \\ u(0, t) = u(\pi, t) = 0 & \text{for } t > 0 \\ u(x, 0) = x^2(\pi - x) & \text{for } x \in (0, \pi). \end{cases}$$

Further, show that your solution obeys $\int_0^\pi u(x, t) dx = 8 \sum_{k \text{ odd}} \frac{1}{k^4} e^{-k^2 t}$. (7 points)

Solution. We will again use the heat kernel and the representation formula, except this time the solution will not be elementary:

$$\begin{aligned} u(x, t) &= 0 - 0 + \int_{[0, \pi]} y^2(\pi - y) \frac{1}{\pi} H_{[0, 1]} \left(\frac{x}{\pi}, \frac{y}{\pi}, \frac{t}{\pi^2} \right) dy \\ &= \frac{2}{\pi} \sum_{k=1}^{\infty} e^{-k^2 t} \sin(kx) \int_{[0, \pi]} y^2(\pi - y) \sin(ky) dy. \end{aligned}$$

The integral can be computed by repeated integration by parts and is $-2\pi k^{-3}(1 + 2 \cos k\pi)$. Hence

$$u(x, t) = -4 \sum_{k=1}^{\infty} k^{-3} (1 + 2 \cos k\pi) e^{-k^2 t} \sin(kx).$$

This is the solution. We can easily compute its value numerically because the series is so rapidly convergent. But we see that it is quite difficult to understand the overall shape. For example, what is the highest temperature of the rod at any given time? Despite the initial condition being positive, it is even not immediately clear that the solution is even positive (though it is, this follows because it is an alternating series and so is bounded between its partial sums). Let's split this into even and odd terms:

$$u(x, t) = 4 \sum_{k \text{ odd}} k^{-3} e^{-k^2 t} \sin(kx) - 12 \sum_{k \text{ even}} k^{-3} e^{-k^2 t} \sin(kx).$$

Because $\int_0^\pi \sin kx dx$ is zero if k is even and $2/k$ if k is odd, the result about the total heat in the domain follows. Notice that the total amount of heat is not conserved in this situation, because the ends of the rod are being kept at a constant temperature of zero and so heat is escaping through these ends.