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Introduction to Partial Differential Equations
Exercise sheet 12

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42. The heat kernel on $[0, 1]$.

(Exercise 4.23 from the lecture script)

(a) Show the final step in the calculation of the heat kernel $H_{[0,1]}$: *(2 points)*

$$\sum_{k=1}^{\infty} e^{-\pi^2 k^2 t}(\cos(k\pi(x-y)) - \cos(k\pi(x+y))) = \frac{1}{2}\Theta(\tfrac{x-y}{2}, \pi i t) - \frac{1}{2}\Theta(\tfrac{x+y}{2}, \pi i t)$$

(b) Let $\mathcal{A}$ be the space of all continuous functions on $\mathbb{R}$ with the following properties:

$$f(n+x) = \begin{cases} f(x) & \text{for even } n \in 2\mathbb{Z} \text{ and } x \in \mathbb{R} \\ -f(1-x) & \text{for odd } n \in 2\mathbb{Z}+1 \text{ and } x \in \mathbb{R}. \end{cases}$$

Show that the functions in $\mathcal{A}$ vanish at $\mathbb{Z}$ and that $\mathcal{A}$ contains all continuous odd and periodic functions with period 2. *(1 point)*

(c) Show that for any Schwartz function f on $\mathbb{R}$ the following series converges to a smooth functions $\tilde{f}$ in $\mathcal{A}$: *(2 points)*

$$\tilde{f}(x) = \sum_{n\in\mathbb{Z}} f(2n+x) - \sum_{n\in\mathbb{Z}} f(2n-x).$$

(d) Show for any $h \in \mathcal{A}$, that the solutions of the heat equation with initial value h is for all $t > 0$ a smooth function in $\mathcal{A}$. Conclude form this that the following sum has the properties of the Heat kernel of $[0, 1]$: *(3 points)*

$$\sum_{n\in\mathbb{Z}} \Phi(x+2n-y, t) - \sum_{n\in\mathbb{Z}} \Phi(x+2n+y, t).$$

(e) Show the relation

$$H_{[0,1]}(x, y, t) = \sum_{n\in\mathbb{Z}} \Phi(x+2n-y, t) - \sum_{n\in\mathbb{Z}} \Phi(x+2n+y, t),$$

where the left hand side the heat kernel in terms of theta functions as given in the lecture script. *(2 bonus points)*

43. Scaling the heat kernel.

Find the heat kernel of

(a) $\mathbb{R}/c\mathbb{R}$ with $c > 0$, *(2 points)*

(b) $[a, b]$ with $-\infty < a < b < \infty$. *(2 points)*

44. Some like it hot.

Find the solution $u : (0, \pi) \times \mathbb{R}^+ \to \mathbb{R}$ of the initial and boundary value problem:

$$\begin{cases} \dot{u} - 7\partial_{xx}u = 0 & \text{for } x \in (0,\pi),\ t > 0 \\ u(0,t) = u(\pi,t) = 0 & \text{for } t > 0 \\ u(x,0) = 3\sin(2x) - 6\sin(5x) & \text{for } x \in (0,\pi). \end{cases}$$

(6 points)

45. Out of the frying pan, into the fire.

Find the solution $u : (0, \pi) \times \mathbb{R}^+ \to \mathbb{R}$ of the initial and boundary value problem:

$$\begin{cases} \dot{u} - \partial_{xx}u = 0 & \text{for } x \in (0,\pi),\ t > 0 \\ u(0,t) = u(\pi,t) = 0 & \text{for } t > 0 \\ u(x,0) = x^2(\pi - x) & \text{for } x \in (0,\pi). \end{cases}$$

Further, show that your solution obeys $\displaystyle\int_0^\pi u(x,t)\,\mathrm{d}x = 8\sum_{k \text{ odd}} \frac{1}{k^4} e^{-k^2 t}$. *(7 points)*