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Introduction to Partial Differential Equations
Exercise sheet 11

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39. Maximum Heat.

In this exercise we extend Theorem 3.13 to the heat equation. Suppose that Ω is a bounded domain and u is a twice continuously differentiable function with $\dot{u} - \Delta u \leq 0$ on Ω_T (by which we mean on the interior). By compactness, u obtains a maximum on $\overline{\Omega_T}$. We will prove that the maximum is obtained on the parabolic boundary.

Recall from following piece of linear algebra from Theorem 3.13: if the Hessian of v is negative semidefinite then $\Delta v \leq 0$. This is similar to Exercise 29.

(a) Suppose first that $\dot{u} - \Delta u < 0$ on Ω_T. Let $(x_0, t_0) \in \operatorname{int} \Omega_T$ be a maximum of u. Why do we know that $\dot{u}(x_0, t_0) = 0$? Consider $v(x) = u(x, t_0)$ and argue that we have a contradiction (apply the second derivative test to v). *(2 points)*

(b) Continue the supposition that $\dot{u} - \Delta u < 0$ on Ω_T. Suppose now that the maximum occurs at (x_0, T). Argue that this also leads to a contradiction. *(2 points)*

(c) Therefore, if $\dot{u} - \Delta u < 0$ on Ω_T then the maximum is obtained on the parabolic boundary. Suppose now that $\dot{u} - \Delta u \leq 0$. Set $u_\varepsilon(x, t) = u(x, t) - \varepsilon t$. Use an argument similar to the one in Theorem 3.13 to show that u takes its maximum on the parabolic boundary. *(2 points)*

(d) *A monotonicity property* For $j \in \{1, 2\}$ let $f_j : \Omega \times (0, T) \to \mathbb{R}$, $h_j : \Omega \to \mathbb{R}$, and $g_j : \partial\Omega \times [0, T] \to \mathbb{R}$ be smooth functions, and likewise let $u_j : \Omega \times (0, T) \to \mathbb{R}$ be smooth functions with continuous extensions to the boundary that satisfy

$$\begin{cases} \dot{u}_j - \Delta u_j = f_j & \text{on } \Omega \times (0, T) \\ u_j(x, 0) = h_j(x) & \text{on } \Omega \\ u_j = g_j & \text{on } \partial\Omega \times [0, T]. \end{cases}$$

Suppose further that $f_1 \leq f_2$, $g_1 \leq g_2$, and $h_1 \leq h_2$. Show in this case that $u_1 \leq u_2$ as well. *(2 points)*

Solution.

(a) At an interior maximum point, the tangent plane must be horizontal. Therefore $\dot{u} = 0$. Consider the function $v(x) = u(x, t_0)$. Then x_0 is a maximum of v, so its Hessian must be negative semidefinite. On the other hand $0 - \Delta v(x_0) < 0$. This is a contradiction.

(b) In this case, we are in the (ordinary) boundary of Ω_T, so we cannot say that the tangent plane is horizontal. However, we can be sure that $\dot{u} \geq 0$. If $\dot{u} < 0$ this means we could reverse time a little and increase the value of u. But by assumption we are at a maximum. Taking the limit of the PDE onto the boundary gives $v(x_0) > \dot{u} \geq 0$. This again contradicts that $v(x_0)$ is a maximum of v.

(c) We see that $\dot{u}_\varepsilon = \dot{u} - \varepsilon$ and $\Delta u_\varepsilon = \Delta u$. Therefore $\dot{u}_\varepsilon - \Delta u_\varepsilon = \dot{u} - \Delta u - \varepsilon \leq -\varepsilon < 0$. Hence we know that the maximum of u_ε is obtain on the parabolic boundary for all $\varepsilon > 0$. We conclude

$$\sup_{\Omega_T} u - \varepsilon T \leq \sup_{\Omega_T} u - \varepsilon t = \sup_{\Omega_T} u_\varepsilon = \max_{\partial\Omega_T} u_\varepsilon \leq \max_{\partial\Omega_T} u.$$

Taking the limit $\varepsilon \to 0$ gives the result.

(d) Set $v = u_1 - u_2$. Because $(\partial_t - \Delta)v = f_1 - f_2 \leq 0$, we know that v obtains its maximum on the parabolic boundary. From the boundary data, we also know that v is non-positive on Ω and $\partial\Omega \times [0, T]$. Hence v is non-positive on the whole domain. $v \leq 0$ implies $u_1 \leq u_2$.

40. Heat death of the universe.

First a corollary to Theorem 4.3:

(a) Suppose that $h \in C_b(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and u is defined as in Theorem 4.3. Show

$$\sup_{x \in \mathbb{R}^n} |u(x,t)| \leq \frac{1}{(4\pi t)^{n/2}} \|h\|_{L^1}.$$

(2 points)

The above corollary shows how solutions to the heat equation on $\mathbb{R}^n \times \mathbb{R}^+$ with such initial conditions behave: they tend to zero as $t \to \infty$. Physically this is because if $h \in L^1$ then there is a finite amount of total heat, which over time becomes evenly spread across the plane.

On open and bounded domains $\Omega \subset \mathbb{R}^n$ we can have different behaviour, due to the boundary conditions holding the temperature steady. In this question we determine the long time behaviour of solutions u to the heat equation on open and bounded sets Ω with $u(x,t) = 0$ on $\partial\Omega \times \mathbb{R}^+$ and $u(x, 0) = h(x)$.We claim $u \to 0$ as $t \to \infty$.

(b) Let l_m be the function from Theorem 4.3 that solves heat equation on $\mathbb{R}^n$ with $l_m(x,0) = mk(x)$ for m a constant and $k : \mathbb{R}^n \to [0,1]$ a smooth function of compact support such that $k|_\Omega \equiv 1$. Why must k exist? Why does $l_m \to 0$ as $t \to \infty$? What boundary conditions on Ω does it obey? *(3 points)*

(c) Use the monotonicity property to show that u tends to zero. *(2 points)*

Hint. Consider $a = \sup_{x\in\Omega} |u(x,0)|$.

Solution.

(a) From the formula in Theorem 4.3 and the definition of the heat kernel

$$|u(x,t)| \leq \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} |h(y)|\, dy \leq \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} |h(y)|\, dy = \frac{1}{(4\pi t)^{n/2}} \|h\|_{L^1}.$$

(b) Since Ω is bounded, it is contained in a ball $B(0, R)$. Choose k to be a hat function that is identically 1 on $B(0, R)$ and zero outside $B(0, 2R)$. We have shown how to construct such hat function in the tutorials.

$mk(x)$ is a smooth function of compact support, so it is continuous, bounded, and has finite L^1 norm. Therefore Part (a) applies to it.

We can see directly from the integral that l_m is non-negative, in particular on $\partial\Omega \times \mathbb{R}^+$. And at time zero, we know from Theorem 4.3 that $l_m(x, 0) = mk(x) \equiv m$ on $x \in \Omega$.

(c) Let $a = \sup_{x\in\Omega} |u(x, 0)| = \sup_{x\in\Omega} |h(x)|$. By definition then $l_{-a}(x, 0) \leq u(x, 0) \leq l_a(x, 0)$ on $\Omega \times \{0\}$. On the parabolic boundary $(x, t) \in \partial\Omega \times \mathbb{R}^+$ we see that $l_{-a}(x, t) \leq u(x, t) = 0 \leq l_a(x, t)$. By the monotonicity property it follows that $l_{-a}(x, t) \leq u(x, t) \leq l_a(x, t)$ for all points. The squeeze theorem then shows that $u \to 0$ as $t \to \infty$.

41. The Fourier transform.

Recall that the Fourier transform of a function $h(x) : \mathbb{R}^n \to \mathbb{R}$ is defined in Section 4.6 to be a function $\hat{h}(k) : \mathbb{R}^n \to \mathbb{R}$ given by

$$\hat{h}(k) = \int_{\mathbb{R}^n} e^{-2\pi i k\cdot y} h(y)\, dy.$$

Lemma 4.20 shows that it is well-defined for Schwartz functions.

(a) Give the definition of a Schwartz function. *(1 point)*

(b) Argue that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \exp(-x^2)$ is a Schwartz function. *(1 point)*

(c) Show that the Fourier transform of $\exp(-A^2x^2)$ for a constant $A > 0$ is $\sqrt{\pi}A^{-1} \exp(-\pi^2k^2A^{-2})$. You may use that $\int_{ai+\mathbb{R}} \exp(-x^2)\, dx = \sqrt{\pi}$ for any a. *(2 points)*

(d) Show that $\widehat{\partial_j f}(k) = 2\pi i k_j \hat{f}(k)$ for Schwartz functions $f : \mathbb{R}^n \to \mathbb{R}$. *(2 points)*

(e) If $u : \mathbb{R} \times \mathbb{R}^+$ is a solution to the heat equation, we can apply a Fourier transform in the space coordinate to get a function $\hat{u}(k, t)$. Show that this function obeys

$$\frac{\partial \hat{u}}{\partial t} + 4\pi^2k^2\hat{u} = 0.$$

Solve this ODE in the time variable. *(2 points)*

(f) Suppose that we have the initial condition $u(x, 0) = h(x)$ for $x \in \mathbb{R}$ for a Schwartz function h. Then $\hat{u}(k, 0) = \hat{h}(k)$. Apply the inverse Fourier transformation to rederive the solution given in Theorem 4.3. *(2 points)*

Solution.

(a) A Schwartz function is a smooth functions whose partial derivatives (of all orders) decay faster than the reciprocal of any polynomial. For such functions, for all multi-indices α and $k \in \mathbb{N}$

$$\lim_{|x|\to\infty} |x|^k \partial^\alpha f(x) = 0.$$

Clearly because this decays to zero, $\sup |x|^k |\partial^\alpha f(x)|$ exists. Conversely, if this supremum exists, then

$$\lim_{|x|\to\infty} |x|^k |\partial^\alpha f(x)| = \lim_{|x|\to\infty} |x|^{-1} |x|^{k+1} |\partial^\alpha f(x)| \le \lim_{|x|\to\infty} |x|^{-1} \sup |x|^{k+1} |\partial^\alpha f(x)| = 0.$$

Therefore these two conditions are equivalent. The second version, $\sup |x|^k |\partial^\alpha f(x)| < \infty$, is often more useful.

(b) This is a function of one variable, so we do not need to use multi-indices.

$$\begin{aligned} f &= e^{-x^2}, \\ f' &= -2xe^{-x^2} \\ f'' &= -2e^{-x^2} + 4x^2 e^{-x^2}. \end{aligned}$$

It is clear that higher derivatives derivative have the form $P(x)e^{-x^2}$ where $P(x)$ is a polynomial. The exponential terms are dominant for large $|x|$, so $f^{(n)}$ decays faster than any polynomial. Thus f is Schwartz.

(c) We could just proceed by direct computation:

$$\begin{aligned} \hat{f}(k) &= \int_{\mathbb{R}} \exp(-2\pi iky) \exp(-A^2 y^2)\, dy \\ &= \int_{\mathbb{R}} \exp(-2\pi iky - A^2 y^2)\, dy \\ &= \int_{\mathbb{R}} \exp -A^2 \Big([\pi ik A^{-2}]^2 + 2\pi ikA^{-2} y + y^2 - [\pi ikA^{-2}]^2 \Big)\, dy \\ &= \exp -[\pi kA^{-1}]^2 \int_{\mathbb{R}} \exp -[\pi ikA^{-1} + Ay]^2\, dy \\ &= \exp -[\pi kA^{-1}]^2 \int_{\pi ikA^{-1}+\mathbb{R}} \exp(-z^2)\, A^{-1}\, dz \\ &= \frac{\sqrt{\pi}}{A} \exp(-\pi^2 k^2 A^{-2}). \end{aligned}$$

In the last step we used the result allowed by the question that $\int_{ai+\mathbb{R}} \exp(-x^2)\, dx = \sqrt{\pi}$ for any a. This was covered in the lecture more or less, but let us present two additional arguments for it.

First however, we should address the real case $\int_{-\infty}^{\infty} \exp(-x^2)\, dx = \sqrt{\pi}$. As best as I can tell, de Moivre was the first to prove something equivalent to this calculation. In 1733 he showed that the binomial distribution in probability theory could be approximated by an exponential term

$$\binom{n}{\frac{n}{2}+d} 0.5^n \approx Ce^{-2d^2/n}$$

and gave a numerical value for the constant. This is equivalent an approximation for the factorial, to which Stirling is given credit for calculating exact constant

$$\lim_{n\to\infty} \frac{n!}{n^{n+0.5}e^{-n}} = \sqrt{2\pi}$$

using the Wallis product for π (1656)

$$\frac{\pi}{2} = \left(\frac{2}{1}\cdot\frac{2}{3}\right)\left(\frac{4}{3}\cdot\frac{4}{5}\right)\left(\frac{6}{5}\cdot\frac{6}{7}\right)\cdots$$

De Moivre and Stirling were contemporaries, Stirling had just written a big book on infinite series, and de Moivre (according to a quote I've seen) gives Stirling credit, but it's not entirely clear to me how involved Stirling actually was. Perhaps de Moivre was just being modest. My sources for these historical remarks are Lee, Pearson, 1924, and Stahl, 2006.

In any case, since the sum of probabilities must add to 1, and the sum can be approximated by an integral, this implies the result. The first person to explicitly state the result was Laplace in 1774. Gauss' name seems be attached to this integral because of his role in popularising e^{-x^2} as a model of measurement errors in astronomy and other sciences. The modern, most common, proof is due to Poisson, simplifying a method of his PhD advisor Laplace. It's 'one from the book':

Poisson's argument in the real case: Consider the square of the integral and apply the Fubini theorem to turn it into an area integral

$$I^2 = \left(\int_{-\infty}^{\infty} \exp(-x^2)\,dx\right)\left(\int_{-\infty}^{\infty} \exp(-y^2)\,dy\right) = \int_{\mathbb{R}^2} \exp(-x^2-y^2)\,dx\,dy.$$

Now make a substitution into polar coordinates

$$I^2 = \int_0^{2\pi}\int_0^{\infty} \exp(-r^2)\,r\,dr\,d\theta = \int_0^{2\pi} \Big[-\frac{1}{2}\exp(-r^2)\Big]_0^{\infty}\,d\theta = \pi.$$

Voilà!

Argument from complex analysis: For the integral along a line in the complex plane, it is of course natural to use a result of complex analysis. Consider

$$g(k) = \int_{\mathbb{R}} \exp -[\pi ikA^{-1} + Ay]^2\,dy.$$

This is an analytic function in the variable $k \in \mathbb{C}$. For all $k \in i\mathbb{R}$, the transformation $z = \pi ikA^{-1} + Ay$ is a real transformation, so it does indeed reduce to $g(k) = A^{-1}\int_{\mathbb{R}}\exp(-z^2)\,dz = A^{-1}\sqrt{\pi}$. The unique continuation property says that two (in this case complex) analytic functions that agree on a sequence and its limit must agree everywhere. Think of the right hand side as the constant function $A^{-1}\sqrt{\pi}$, which is analytic. Hence $g(k) = A^{-1}\sqrt{\pi}$ for all k, not just imaginary k.

Argument from lecture notes: In the lecture notes, Prof Schmidt gave an very concrete argument using power series. We begin with the observation from our working above that

$$\int_{\mathbb{R}} e^{-(\pi ikA^{-1}+Ay)^2+(\pi ikA^{-1})^2}\,dy = \int_{\mathbb{R}} e^{-2\pi iky-A^2y^2}\,dy.$$

Let us then investigate the following integral for real values of ω. By the same algebraic manipulations, we have

$$\int_{\mathbb{R}} e^{-(\omega+Ay)^2+\omega^2}\,dy = \int_{\mathbb{R}} e^{-2\omega Ay-A^2y^2}\,dy.$$

The left hand side is easy to manipulate

$$\int_{\mathbb{R}} e^{-(\omega+Ay)^2+\omega^2}\,dy = e^{\omega^2}\int_{\mathbb{R}} e^{-(\omega+Ay)^2}\,dy = e^{\omega^2}\int_{\mathbb{R}} e^{-z^2}\,A^{-1}dz = A^{-1}\sqrt{\pi}e^{\omega^2} = \sum_{l=0}^{\infty}\frac{A^{-1}\sqrt{\pi}}{l!}\omega^{2l}$$

On the right hand side, we expand this into a power series in ω:

$$\int_{\mathbb{R}} e^{-2\omega Ay-A^2y^2}\,dy = \int_{\mathbb{R}} e^{-A^2y^2}\sum_{l=0}^{\infty}\frac{(-2\omega Ay)^l}{l!}\,dy = \sum_{l=0}^{\infty}\left(\int_{\mathbb{R}} e^{-A^2y^2}\frac{(-2Ay)^l}{l!}\,dy\right)\omega^l$$

These two power series are equal, so their coefficients must be equal. In other words

$$\int_{\mathbb{R}} e^{-A^2y^2}\frac{(-2Ay)^l}{l!}\,dy = \begin{cases}\frac{A^{-1}\sqrt{\pi}}{(l/2)!} & l \text{ even}\\ 0 & l \text{ odd}\end{cases}$$

We can now return to the imaginary case. Again, making the power series expansion in k

$$\begin{aligned}\int_{\mathbb{R}} e^{-2\pi iky-A^2y^2}\,dy &= \int_{\mathbb{R}} e^{-A^2y^2}\sum_{l=0}^{\infty}\frac{(-2\pi ikA^{-1}Ay)^l}{l!}\,dy\\ &= \sum_{l=0}^{\infty}\left(\int_{\mathbb{R}} e^{-A^2y^2}\frac{(-2Ay)^l}{l!}\,dy\right)(\pi ikA^{-1})^l\\ &= \sum_{l=0}^{\infty}\frac{A^{-1}\sqrt{\pi}}{l!}(\pi ikA^{-1})^{2l}\\ &= A^{-1}\sqrt{\pi}\sum_{l=0}^{\infty}\frac{(-\pi^2k^2A^{-2})^l}{l!}\\ &= A^{-1}\sqrt{\pi}\exp(-\pi^2k^2A^{-2}).\end{aligned}$$

This also gives the result. Note here that at the core of the argument was "two equal power series have equal coefficients". This is how one proves that analytic functions have the unique continuation property. So the above two proofs are not as different as they first appear.

Argument from differentiating: I've saved the simplest method until last. Consider the function $g(k)$ again and differentiate

$$\begin{aligned}g(k) &= \int_{\mathbb{R}} e^{-[\pi ikA^{-1}+Ay]^2}\,dy\\ g'(k) &= \int_{\mathbb{R}} -2[\pi ikA^{-1}+Ay]\times\pi iA^{-1}\times e^{-[\pi ikA^{-1}+Ay]^2}\,dy\\ &= \pi iA^{-2}\int_{\mathbb{R}} -2[\pi ikA^{-1}+Ay]\times A\times e^{-[\pi ikA^{-1}+Ay]^2}\,dy\\ &= \pi iA^{-2}\int_{\mathbb{R}}\frac{\partial}{\partial y}e^{-[\pi ikA^{-1}+Ay]^2}\,dy = \pi iA^{-2}e^{-[\pi ikA^{-1}+Ay]^2}\Big|_{y=-\infty}^{y=\infty} = 0.\end{aligned}$$

Thus $g(k)$ is a constant function, and is equal to $g(0)$. But this is then exactly the real integral that we know the value of.

(d) Let's do this for $j = n$ to make the notation easier:

$$\begin{aligned}
\widehat{\partial_n f}(k) &= \int_{\mathbb{R}^n} e^{-2\pi i k\cdot y}\partial_n f(y)\,dy = \int_{\mathbb{R}} e^{-2\pi i k_1 y_1}\cdots\int_{\mathbb{R}} e^{-2\pi i k_n y_n}\partial_n f(y)\,dy_n\ldots dy_1 \\
&= \int_{\mathbb{R}} e^{-2\pi i k_1 y_1}\cdots\left[e^{-2\pi i k_n y_n}f(y)\Big|_{y_n=-\infty}^{y_n=\infty} + 2\pi i k_n\int_{\mathbb{R}} e^{-2\pi i k_n y_n}f(y)\,dy_n\right]dy_{n-1}\ldots dy_1 \\
&= 2\pi i k_n\int_{\mathbb{R}} e^{-2\pi i k_1 y_1}\cdots\left[\int_{\mathbb{R}} e^{-2\pi i k_n y_n}f(y)\,dy_n\right]dy_{n-1}\ldots dy_1 \\
&= 2\pi i k_n\hat{f}(k).
\end{aligned}$$

We know that f vanishes at infinity because it is a Schwartz function.

(e) From the previous question, we know how to calculate the Fourier transform of the Laplacian

$$\widehat{\sum\partial_j^2 u} = \sum\widehat{\partial_j\partial_j u} = \sum 2\pi i k_j\widehat{\partial_j u} = \sum 4\pi^2 i^2 k_j^2\hat{u} = -4\pi^2|k|^2\hat{u}.$$

In one dimension, the Laplacian is just the second derivative and the vector length simplifies too $|k|^2 = k^2$. On the other hand, we can pass the derivative with respect to time through the integral, so $\widehat{\partial_t u} = \partial_t\hat{u}$.

Now, for fixed k, we have a first-order linear ODE in t. It has the solution

$$\hat{u}(k,t) = \hat{u}(k,0)\exp(-4\pi^2k^2t).$$

(f)

$$\begin{aligned}
u(x,t) &= \int_{\mathbb{R}} e^{2\pi i k x}\hat{u}(k,t)\,dk = \int_{\mathbb{R}} e^{2\pi i k x}\,\hat{h}(k)\exp(-4\pi^2k^2t)\,dk \\
&= \int_{\mathbb{R}} e^{2\pi i k x}e^{-4\pi^2k^2t}\int_{\mathbb{R}} e^{-2\pi i k y}h(y)\,dy\,dk \\
&= \int_{\mathbb{R}} h(y)\int_{\mathbb{R}} e^{-2\pi i(y-x)k-4\pi^2tk^2}\,dk\,dy
\end{aligned}$$

But notice that the inner integral has the exact same form as the integral in part (c). Therefore we arrive at

$$u(x,t) = \int_{\mathbb{R}^n} h(y)\frac{\sqrt{\pi}}{2\pi\sqrt{t}}\exp\frac{-\pi^2(y-x)^2}{4\pi^2t}\,dy. = \int_{\mathbb{R}^n} h(y)\frac{1}{\sqrt{4\pi t}}\exp\frac{-(y-x)^2}{4t}\,dy.$$

As Prof Schmidt pointed out to me, in the script we use Theorem 4.3 and the heat kernel to find the inverse Fourier transform in Lemma 4.20. If you instead begin with the inverse Fourier transform, as we do in this question, then you can derive the heat kernel.