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Introduction to Partial Differential Equations
Exercise sheet 11

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39. Maximum Heat.

In this exercise we extend Theorem 3.13 to the heat equation. Suppose that u is a twice continuously differentiable function with $\dot{u} - \Delta u \leq 0$ on a bounded domain Ω. By compactness, u obtains a maximum on $\overline{\Omega}$. We will prove that the maximum is obtained on the parabolic boundary.

Recall from following piece of linear algebra from Theorem 3.13: if the Hessian of v is negative semidefinite then $\Delta v \leq 0$. This is similar to Exercise 29.

(a) Suppose first that $\dot{u} - \Delta u < 0$ on Ω. Let $(x_0, t_0) \in \Omega_T$ be a maximum of u. Why do we know that $\dot{u}(x_0, t_0) = 0$? Consider $v(x) = u(x, t_0)$ and argue that we have a contradiction (apply the second derivative test to v). *(2 points)*

(b) Continue the supposition that $\dot{u} - \Delta u < 0$ on Ω. Suppose now that the maximum occurs at (x_0, T). Argue that this also leads to a contradiction. *(2 points)*

(c) Therefore, if $\dot{u} - \Delta u < 0$ on Ω then the maximum is obtained on the parabolic boundary. Suppose now that $\dot{u} - \Delta u \leq 0$. Set $u_\varepsilon(x, t) = u(x, t) - \varepsilon t$. Use an argument similar to the one in Theorem 3.13 to show that u takes its maximum on the parabolic boundary. *(2 points)*

(d) *A monotonicity property* For $j \in \{1, 2\}$ let $f_j : \Omega \times (0, T) \to \mathbb{R}$, $h_j : \Omega \to \mathbb{R}$, and $g_j : \partial\Omega \times [0, T] \to \mathbb{R}$ be smooth functions, and likewise let $u_j : \Omega \times (0, T) \to \mathbb{R}$ be smooth functions with continuous extensions to the boundary that satisfy

$$\begin{cases} \dot{u}_j - \Delta u_j = f_j & \text{on } \Omega \times (0, T) \\ u_j(x, 0) = h_j(x) & \text{on } \Omega \\ u_j = g_j & \text{on } \partial\Omega \times [0, T]. \end{cases}$$

Suppose further that $f_1 \leq f_2$, $g_1 \leq g_2$, and $h_1 \leq h_2$. Show in this case that $u_1 \leq u_2$ as well. *(2 points)*

40. Heat death of the universe.

First a corollary to Theorem 4.3:

(a) Suppose that $h \in C_b(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and u is defined as in Theorem 4.3. Show

$$\sup_{x \in \mathbb{R}^n} |u(x, t)| \leq \frac{1}{(4\pi t)^{n/2}} \|h\|_{L^1}.$$

(2 points)

The above corollary shows how solutions to the heat equation on $\mathbb{R}^n \times \mathbb{R}^+$ with such initial conditions behave: they tend to zero as $t \to \infty$. Physically this is because if $h \in L^1$ then there is a finite amount of total heat, which over time becomes evenly spread across the plane.

On open and bounded domains $\Omega \subset \mathbb{R}^n$ we can have different behaviour, due to the boundary conditions holding the temperature steady. In this question we determine the long time behaviour of solutions u to the heat equation on open and bounded sets Ω with $u(x,t) = 0$ on $\partial\Omega \times \mathbb{R}^+$ and $u(x,0) = h(x)$.We claim $u \to 0$ as $t \to \infty$.

(b) Let l_m be the function from Theorem 4.3 that solves heat equation on $\mathbb{R}^n$ with $l_m(x,0) = mk(x)$ for m a constant and $k : \mathbb{R}^n \to [0,1]$ a smooth function of compact support such that $k|_\Omega \equiv 1$. Why must k exist? Why does $l_m \to 0$ as $t \to \infty$? What boundary conditions on Ω does it obey? *(3 points)*

(c) Use the monotonicity property to show that u tends to zero. *(2 points)*

Hint. Consider $a = \sup_{x\in\Omega} |u(x,0)|$.

41. The Fourier transform.

Recall that the Fourier transform of a function $h(x) : \mathbb{R}^n \to \mathbb{R}$ is defined in Section 4.6 to be a function $\hat{h}(k) : \mathbb{R}^n \to \mathbb{R}$ given by

$$\hat{h}(k) = \int_{\mathbb{R}^n} e^{-2\pi i k\cdot y} h(y)\, dy.$$

Lemma 4.20 shows that it is well-defined for Schwartz functions.

(a) Give the definition of a Schwartz function. *(1 point)*

(b) Argue that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \exp(-x^2)$ is a Schwartz function. *(1 point)*

(c) Show that the Fourier transform of $\exp(-A^2x^2)$ for a constant $A > 0$ is $\sqrt{\pi}A^{-1}\exp(-\pi^2k^2A^{-2})$. You may use that $\int_{ai+\mathbb{R}} \exp(-x^2)\,dx = \sqrt{\pi}$ for any a. *(2 points)*

(d) Show that $\widehat{\partial_j f}(k) = 2\pi i k_j \hat{f}(k)$ for Schwartz functions $f : \mathbb{R}^n \to \mathbb{R}$. *(2 points)*

(e) If $u : \mathbb{R} \times \mathbb{R}^+$ is a solution to the heat equation, we can apply a Fourier transform in the space coordinate to get a function $\hat{u}(k,t)$. Show that this function obeys

$$\frac{\partial \hat{u}}{\partial t} + 4\pi^2k^2\hat{u} = 0.$$

Solve this ODE in the time variable. *(2 points)*

(f) Suppose that we have the initial condition $u(x,0) = h(x)$ for $x \in \mathbb{R}$ for a Schwartz function h. Then $\hat{u}(k,0) = \hat{h}(k)$. Apply the inverse Fourier transformation to rederive the solution given in Theorem 4.3. *(2 points)*