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Introduction to Partial Differential Equations
Exercise sheet 10

8th November, 2022

36. Special solutions of the heat equation.

(a) Solutions of PDEs that are constant in the time variable are called "steady-state" solutions. Describe steady-state solutions of the inhomogeneous heat equation. *(1 point)*

(b) Consider the heat equation $\dot{u} - \Delta u = 0$ on $\mathbb{R}^n \times \mathbb{R}^+$ with smooth initial condition $u(x, 0) = h(x)$. Suppose that the Laplacian of h is a constant. Show that there is a solution whose time derivative is constant. *(1 point)*

(c) Consider "translational solutions" to the heat equation on $\mathbb{R} \times \mathbb{R}^+$ (ie $n = 1$). These are solutions of the form $u(x, t) = F(x - bt)$. Find all such solutions. *(2 points)*

(d) If u is a solution to the heat equation, show for every $\lambda \in \mathbb{R}$ that $u_\lambda(x, t) := u(\lambda x, \lambda^2 t)$ is also a solution to the heat equation. *(2 points)*

Solution.

(a) Steady-state solutions are defined by $\dot{u} = 0$. The heat equation then reduces to a Poisson equation: $0 - \Delta u = f$.

(b) If the time derivative of u is constant, then it follows that $u(x, t) = u_0(x) + u_1 t$. From the initial condition we must have $u_0 = h$. Putting this into the heat equation gives

$$u_1 - \Delta h = 0.$$

So choose the constant $u_1 = \Delta h$.

(c) Putting this into the heat equation gives

$$-bF' - F'' = 0.$$

Integrating gives

$$bF + F' = C.$$

This has the solution $F(y) = C + A \exp(-by)$. So we get the solution $u(x, t) = C + A \exp(-bx + b^2 t)$.

(d) This follows because $\partial_t(u(\lambda x, \lambda^2 t)) = \lambda^2 \dot{u}(\lambda x, \lambda^2 t)$ and $\partial_j^2(u(\lambda x, \lambda^2 t)) = \lambda^2 (\partial_j^2 u)(\lambda x, \lambda^2 t)$.

37. One step at a time.

Prove the following identity for the fundamental solution in one dimension ($n = 1$):

$$\Phi(x, s + t) = \int_{\mathbb{R}} \Phi(x - y, t)\Phi(y, s)\, dy.$$

Interpret this equation in the context of Theorem 4.3. *(4 points)*

Hint. You may use without proof that

$$\int_{\mathbb{R}} \exp(-A + By - Cy^2)\, dy = \sqrt{\frac{\pi}{C}} \exp\left(\frac{B^2}{4C} - A\right).$$

Solution.

$$\begin{aligned}
\int_{\mathbb{R}} \Phi(x-y,t)\Phi(y,s)\, dy &= \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}\sqrt{4\pi s}} \exp\left(-\frac{(x-y)^2}{4t} - \frac{y^2}{4s}\right) dy \\
&= \frac{1}{\sqrt{4\pi t}\sqrt{4\pi s}} \int_{\mathbb{R}} \exp\left(-\frac{x^2}{4t} + \frac{x}{2t}y - \left[\frac{1}{4t} + \frac{1}{4s}\right] y^2\right) dy \\
&= \frac{1}{\sqrt{4\pi t}\sqrt{4\pi s}} \sqrt{\frac{\pi}{\frac{s+t}{4st}}} \exp\left(\frac{x^2}{4t^2} \cdot \frac{1}{4} \cdot \frac{4st}{s+t} - \frac{x^2}{4t}\right) \\
&= \frac{1}{\sqrt{4\pi(s+t)}} \exp\left(-\frac{x^2}{4(s+t)}\right) = \Phi(x, s+t).
\end{aligned}$$

We know that convolution of the initial condition with the fundamental solution over the space coordinates gives the solution to the initial value problem. The fact that we have $s+t$ in the time suggests we should do this twice. So if we begin with the initial condition $h(x)$ and then solve up to time t, we have

$$u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t)h(y)\, dy.$$

If we take $x \mapsto u(x,t)$ as the start of a new initial value problem and then calculate what the solution is at time s we get

$$\begin{aligned}
v(x,s) &= \int_{\mathbb{R}} \Phi(x-y,s)u(y,t)\, dy \\
&= \int_{\mathbb{R}} \Phi(x-y,s) \int_{\mathbb{R}} \Phi(y-z,t)h(z)\, dz\, dy \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Phi(x-y,s)\Phi(y-z,t)\, dy\right) h(z)\, dz \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Phi(x-z-y,s)\Phi(y,t)\, dy\right) h(z)\, dz.
\end{aligned}$$

(We made a shift substitution to get to the last line, but reused the letter y.) On the other hand, if we began with the initial condition $h(x)$ and computed the solution at time $s+t$ we would find

$$u(x,s+t) = \int_{\mathbb{R}} \Phi(x-z,s+t)h(z)\, dz.$$

The relation we just proved says that these two solutions are the same. This shows that the heat equation does not have 'long term memory', it only depends on the immediately prior state, not on anything that happened before that. This is in contrast to its behaviour in space, where the action at one point can immediately affect points infinitely far away. This property is also called the semigroup property because the solution operators $h \mapsto H_t h := \Phi(\cdot,t) * h$ form a semigroup: $H_s H_t = H_{s+t}$.

38. The distribution of heat.

Consider the fundamental solution of the heat equation $\Phi(x,t)$ given in Definition 4.1.

(a) Show that this extends to a smooth function on $\mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\}$. *(2 points)*

(b) Verify that this obeys the heat equation on $\mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\}$. *(2 points)*

We want to show that $\varphi \mapsto H(\varphi) = \int_{\mathbb{R}^n\times\mathbb{R}} \Phi(x,t)\varphi(x,t)\ dx\,dt$ is a distribution. Clearly it is linear. Fix a set $K \subset \mathbb{R}^n \times \mathbb{R}$ and let $\varphi \in C_0^\infty(K)$.

(c) Why must there be a constant $T > 0$ with

$$H(\varphi) = \int_0^T \int_{\mathbb{R}^n} \Phi(x,t)\varphi(x,t)\ dx\,dt\ \ ?$$

(1 point)

(d) Conclude with the help of Lemma 4.2 and Theorem 4.3 that

$$|H(\varphi)| \leq T\,\|\varphi\|_{K,0}.$$

Hence H is a continuous linear functional. *(3 points)*

Finally, we want to show that (in the sense of distributions) $(\partial_t - \Delta)H = \delta$.

(e) Extend Theorem 4.3 to show that

$$\int_{\mathbb{R}^n} \Phi(x-y,t)h(y,s)\ dy \to h(x,s)$$

as $t \to 0$, uniformly in s. *(1 point)*

(f) Hence show that

$$\int_\varepsilon^\infty \int_{\mathbb{R}^n} \Phi(-\partial_t\varphi - \Delta\varphi)\ dy\,dt \to \varphi(0,0)$$

as $\varepsilon \to 0$. *(4 points)*

(g) Prove that as $\varepsilon \to 0$

$$\int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,t)h(y,t)\ dy\,dt \to 0$$

(2 points)

Together these integrals show that

$$(\partial_t - \Delta)H(\varphi) = \left(\int_0^\varepsilon + \int_\varepsilon^\infty\right)\int_{\mathbb{R}^n} \Phi(-\partial_t\varphi - \Delta\varphi)\ dy\,dt = \varphi(0,0) = \delta(\varphi)$$

for all test functions φ. Therefore $(\partial_t - \Delta)H = \delta$ as claimed.

Solution.

(a) For $t > 0$, $\Phi(x,t)$ and all its derivatives have the form $t^{-k}q(x,t)\exp(-x^2/4t)$ for $k \in \mathbb{N}_0$ and q a polynomial. As $t \to 0^+$ for $x \neq 0$, the exponential terms is dominant and forces the expression to zero. For $t < 0$ the heat kernel is identically zero, and so all its derivatives are zero and the limits as $t \to 0^-$ is zero. Thus we have the smooth extension $\Phi(x,0) = 0$ for $x \neq 0$.

(b) By direct calculation, for $t > 0$

$$\begin{aligned}
(4\pi)^{n/2}\partial_t\Phi &= -\frac{n}{2}t^{-n/2-1}e^{-\frac{|x|^2}{4t}} + \frac{|x|^2}{4}t^{-n/2-2}e^{-\frac{|x|^2}{4t}} \\
(4\pi)^{n/2}\partial_j\Phi &= -\frac{x_j}{2}t^{-n/2-1}e^{-\frac{|x|^2}{4t}} \\
(4\pi)^{n/2}\partial_j^2\Phi &= -\frac{1}{2}t^{-n/2-1}e^{-\frac{|x|^2}{4t}} + \frac{x_j^2}{4}t^{-n/2-2}e^{-\frac{|x|^2}{4t}}.
\end{aligned}$$

The appropriate sum gives zero. We see that all derivatives of the function are zero for $t = 0, x \neq 0$. Therefore the heat equation holds there too.

(c) Because φ has compact support, it is zero outside $B(0,R) \times [-T,T]$ for some positive constants R and T. Additionally, we know that Φ is zero for $t < 0$. Therefore the integrand is zero outside $B(0,R) \times [0,T]$ and can be discarded.

(d) First just apply the estimate that bounds φ:

$$|H(\varphi)| \leq \int_0^T \int_{\mathbb{R}^n} \Phi(x,t)\|\varphi(x,t)\|_{K,0}\,dx\,dt = \|\varphi(x,t)\|_{K,0}\int_0^T \int_{\mathbb{R}^n} \Phi(x,t)\,dx\,dt.$$

So it remains to bound the integral of Φ over this region. Consider the function $g(t) := \int_{\mathbb{R}^n} \Phi(x,t)\,dx$. Lemma 4.2 says that $g(t) = 1$ for $t > 0$. Theorem 4.3(iii) says that $g(0) = 1$. This gives

$$\int_0^T \int_{\mathbb{R}^n} \Phi(x,t)\,dx\,dt = \int_0^T 1\,dt = T$$

as the constant.

(e) One only needs to modify one step in the proof of Theorem 4.3: $|h(y,s) - h(x,s)|$ is bounded by twice the supremum of h over space *and time* variables.

(f) We should try to apply integration by parts, in order to move the derivatives from φ to Φ, because we know what Φ is. However the boundary term does not necessarily vanish on the $t = \varepsilon$ plane.

$$\begin{aligned}
-\int_\varepsilon^\infty \int_{\mathbb{R}^n} \Phi\partial_t\varphi - \int_\varepsilon^\infty \int_{\mathbb{R}^n} \Phi\Delta\varphi &= -\left[\int_{\mathbb{R}^n} \Phi\varphi\Big|_{t=\varepsilon}^{t=\infty} - \int_\varepsilon^\infty \int_{\mathbb{R}^n} \partial_t\Phi\varphi\right] - \int_\varepsilon^\infty \int_{\mathbb{R}^n} \Delta\Phi\varphi \\
&= \int_{\mathbb{R}^n} \Phi(x,\varepsilon)\varphi(x,\varepsilon) + \int_\varepsilon^\infty \int_{\mathbb{R}^n} (\partial_t\Phi - \Delta\Phi)\varphi \\
&= \int_{\mathbb{R}^n} \Phi(x,\varepsilon)\varphi(x,\varepsilon) \\
&= \int_{\mathbb{R}^n} \Phi(0 - x,\varepsilon)\varphi(x,\varepsilon).
\end{aligned}$$

The second integral on the second line vanishes due to part (b). We can now apply the previous part to conclude that this limits to $\varphi(0,0)$. Notice the need for uniform convergence, because the second parameter of φ is also being changed by the limit $\varepsilon \to 0$.

(g) We can estimate the norm of h out

$$\left|\int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(x,t)h(x,t)\,dx\,dt\right| \leq \|h\|_\infty \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(x,t)\,dx\,dt. = \|h\|_\infty \int_0^\varepsilon dt = \|h\|_\infty\varepsilon.$$

Clearly this tends to zero.