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Introduction to Partial Differential Equations
Exercise sheet 10

8th November, 2022

36. Special solutions of the heat equation.

(a) Solutions of PDEs that are constant in the time variable are called "steady-state" solutions. Describe steady-state solutions of the inhomogeneous heat equation. *(1 point)*

(b) Consider the heat equation $\dot{u} - \Delta u = 0$ on $\mathbb{R}^n \times \mathbb{R}^+$ with smooth initial condition $u(x, 0) = h(x)$. Suppose that the Laplacian of h is a constant. Show that there is a solution whose time derivative is constant. *(1 point)*

(c) Consider "translational solutions" to the heat equation on $\mathbb{R} \times \mathbb{R}^+$ (ie $n = 1$). These are solutions of the form $u(x, t) = F(x - bt)$. Find all such solutions. *(2 points)*

(d) If u is a solution to the heat equation, show for every $\lambda \in \mathbb{R}$ that $u_\lambda(x, t) := u(\lambda x, \lambda^2 t)$ is also a solution to the heat equation. *(2 points)*

37. One step at a time.

Prove the following identity for the fundamental solution in one dimension ($n = 1$):

$$\Phi(x, s + t) = \int_{\mathbb{R}} \Phi(x - y, t)\Phi(y, s)\, dy.$$

Interpret this equation in the context of Theorem 4.3. *(4 points)*

Hint. You may use without proof that

$$\int_{\mathbb{R}} \exp(-A + By - Cy^2)\, dy = \sqrt{\frac{\pi}{C}} \exp\left(\frac{B^2}{4C} - A\right).$$

38. The distribution of heat.

Consider the fundamental solution of the heat equation $\Phi(x, t)$ given in Definition 4.1.

(a) Show that this extends to a smooth function on $\mathbb{R}^n \times \mathbb{R} \setminus \{(0, 0)\}$. *(2 points)*

(b) Verify that this obeys the heat equation on $\mathbb{R}^n \times \mathbb{R} \setminus \{(0, 0)\}$. *(2 points)*

We want to show that $\varphi \mapsto H(\varphi) = \int_{\mathbb{R}^n \times \mathbb{R}} \Phi(x, t)\varphi(x, t)\, dx\, dt$ is a distribution. Clearly it is linear. Fix a set $K \subset \mathbb{R}^n \times \mathbb{R}$ and let $\varphi \in C_0^\infty(K)$.

(c) Why must there be a constant $T > 0$ with

$$H(\varphi) = \int_0^T \int_{\mathbb{R}^n} \Phi(x, t)\varphi(x, t)\, dx\, dt \ ?$$

(1 point)

(d) Conclude with the help of Lemma 4.2 and Theorem 4.3 that

$$|H(\varphi)| \leq T\, \|\varphi\|_{K,0}.$$

Hence H is a continuous linear functional. *(3 points)*

Finally, we want to show that (in the sense of distributions) $(\partial_t - \Delta)H = \delta$.

(e) Extend Theorem 4.3 to show that

$$\int_{\mathbb{R}^n} \Phi(x-y,t)h(y,s)\;dy \to h(x,s)$$

as $t \to 0$, uniformly in s. *(1 point)*

(f) Hence show that

$$\int_{\varepsilon}^{\infty} \int_{\mathbb{R}^n} \Phi(-\partial_t\varphi - \Delta\varphi)\;dy\,dt \to \varphi(0,0)$$

as $\varepsilon \to 0$. *(4 points)*

(g) Prove that as $\varepsilon \to 0$

$$\int_0^{\varepsilon} \int_{\mathbb{R}^n} \Phi(y,t)h(y,t)\;dy\,dt \to 0$$

(2 points)

Together these integrals show that

$$(\partial_t - \Delta)H(\varphi) = \left(\int_0^{\varepsilon} + \int_{\varepsilon}^{\infty}\right)\int_{\mathbb{R}^n} \Phi(-\partial_t\varphi - \Delta\varphi)\;dy\,dt = \varphi(0,0) = \delta(\varphi)$$

for all test functions φ. Therefore $(\partial_t - \Delta)H = \delta$ as claimed.