

This is a public holiday. Perhaps we will reschedule the tutorial. Either way, you must submit the exercise sheet.

32. Do nothing by halves.

Let $H_1^+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 > 0\}$ be the upper half-space and $H_1^0 = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 = 0\}$ the dividing hyperplane. We call $R_1(x) = (-x_1, x_2, \dots, x_n)$ reflection in the plane H^0 .

(a) **A reflection principle for harmonic functions.** Let $u \in C^2(\overline{H_1^+})$ be a harmonic function that vanishes on H_1^0 . Show that the function $v : \mathbb{R}^n \rightarrow \mathbb{R}$ defined through reflection

$$v(x) = \begin{cases} u(x) & \text{for } x_1 \geq 0 \\ -u(R_1(x)) & \text{for } x_1 < 0 \end{cases}$$

is harmonic.

(4 points)

(b) **Green's function for the half-space.** Show that Green's function for H_1^+ is

$$G(x, y) = \Phi(x - y) - \Phi(R_1(x) - y).$$

(3 points)

(c) **Green's function for the half-ball.** Compute the Green's function for B^+ . (3 points)

Hint. Make use of both the Green's function for the ball 3.20 and part (b).

Solution.

(a) One could try to show directly that v is harmonic. Clearly it is when $x_1 \neq 0$, and it is possible to compute the necessary derivatives when $x_1 = 0$. However, there is a more general method using the uniqueness of the solution to the Dirichlet problem on the ball.

Fix any radius $r > 0$ and consider the ball B_r . Let $g = v|_{\partial B_r}$ be the restriction of this function to the sphere. This is continuous, in particular when $x_1 = 0$. There is a unique solution \tilde{v} to the Laplace equation $\Delta \tilde{v} = 0$ with $\tilde{v}|_{\partial B_r} = g$. We see that $-\tilde{v} \circ R_1$ is also a solution to this equation, thus $\tilde{v} = -\tilde{v} \circ R$. This implies that \tilde{v} vanishes on $B_r \cap H_1^0$.

Now consider $\tilde{v} - u$. This is also a harmonic function on $B_r \cap H_1^+$, and moreover it is identically zero on $\partial(B_r \cap H_1^+)$. The maximum principle says it has to be zero on all of $B_r \cap H_1^+$. Thus $\tilde{v} = u$ on $B_r \cap H_1^+$ and by reflection $\tilde{v} = v$ on B_r . By taking r larger and larger, we see that equality holds for all points of the plane.

(b) Let Φ be the fundamental solution to the Laplace equation. Let $G(x, y)$ be the Green's function for H_1^+ . The required properties are (1) that for any $x \in H_1^+$ the function $G(x, y) - \Phi(x - y)$ is a harmonic function of y and (2) that for any $x \in H_1^+$ we have $\lim_{y \rightarrow H_1^0} G(x, y) = 0$. We saw for the unit ball that the Greens function was a difference of the fundamental solution and its reflection across the boundary of the ball. That way, the two cancelled on the boundary and gave the second property. So let use try

$$G(x, y) = \Phi(x - y) - \Phi(R_1(x) - y).$$

The first property is satisfied because $\Phi(R_1(x) - y)$ is only not harmonic when $y = R(x)$, and for $x \in H_1^+$ this only occurs when $y \in H_1^-$. To show the second property, note that $\Phi(z)$ is radially symmetric. Since $\|R(x) - y\| = \|x - R_1(y)\|$, for any $y \in H^0$ we have

$$G(x, y) = \Phi(x - y) - \Phi(x - R_1(y)) = \Phi(x - y) - \Phi(x - y) = 0.$$

Thus we have shown that G is the Greens function.

- (c) To discuss the Greens function for the half-ball, we should introduce a symbol for inversion in the sphere, $\iota(x) := |x|^{-2}x$. We know from lectures then

$$G_B(x, y) = \Phi(x - y) - \Phi(|x|(\iota(x) - y)).$$

Following the ideas of the previous question, we guess that the Greens function for the half-ball is the reflection of this one

$$G(x, y) = \Phi(x - y) - \Phi(|x|(\iota(x) - y)) - \Phi(R(x) - y) + \Phi(|x|(\iota(R(x)) - y)).$$

If both $x, y \in B^+$ then $\iota(x) - y$, $R(x) - y$, and $|x|(\iota(R(x)) - y)$ are never zero, so $G(x, y) - \Phi(x, y)$ is harmonic. The boundary of B^+ has two parts $\overline{B^0}$ and $\partial B^+ \cap H^+$. If $y \in \overline{B^0}$ then

$$\begin{aligned} G(x, y) &= \Phi(x - y) - \Phi(|x|(\iota(x) - y)) - \Phi(x - R(y)) + \Phi(|x|(\iota(R(x)) - y)) \\ &= \Phi(x - y) - \Phi(|x|(\iota(x) - y)) - \Phi(x - y) + \Phi(|x|(\iota(x) - y)) = 0. \end{aligned}$$

On the other hand, if $y \in \partial B^+ \cap H^+$ is in the hemispherical part, then $\| |x|(\iota(x) - y) \| = \|x - y\|$ as in the lecture notes, but also $\| |x|(\iota(R(x)) - y) \| = \|R(x) - y\|$, so

$$G(x, y) = [\Phi(x - y) - \Phi(|x|(\iota(x) - y))] - [\Phi(R(x) - y) - \Phi(|x|(\iota(R(x)) - y))] = 0.$$

33. It's not easy being green.

Suppose that Ω is a bounded domain. Prove that there is at most one Green's function on Ω .

(3 points)

On the other hand, suppose that Ω has a Green's function G_Ω and that there exists a non-trivial solution to the Dirichlet problem

$$\Delta u = 0, \quad u|_{\partial\Omega} = 0.$$

Construct a second Green's function for Ω .

(2 points)

Solution. Suppose that we have two Green's functions G and G' . Fix $x \in \Omega$. Then

$$y \mapsto G(x, y) - G'(x, y) = [G(x, y) - \Phi(x - y)] - [G'(x, y) - \Phi(x - y)]$$

is harmonic and vanishes on the boundary $\partial\Omega$. By the maximum principle, this must be zero. Hence $G = G'$.

If u is the non-trivial solution, then $G_\Omega(x, y) + u(y)$ is another Green's function.

34. One of a kind.

Let $\Omega \subseteq \mathbb{R}^n$ be an open and connected domain and $u, v \in C^2(\overline{\Omega})$ two harmonic functions. Suppose that there is an open subset $U \subset \Omega$ such that $u = v$ on U . Prove that $u = v$ on Ω using Corollary 3.22 (Harmonic functions are analytic). This is called the *unique continuation property* of harmonic functions. (3 points)

Solution. Because of Corollary 3.22, all harmonic functions are analytic. All analytic functions have the unique continuation property, so there is nothing to show.

But let us recall the argument that analytic functions have the unique continuation property. Consider the difference $w = u - v$, it is also analytic. It is enough to show that this function is identically zero on Ω given that it is zero on U .

Analytic functions are functions that for every point x there is a ball $B(x, r_x)$ such that w is equal to a power series centred at x . If a power series vanishes on an open set, then all its coefficients must be zero. Choose any point $b \in \Omega$ and a point $a \in U$. Connect them with a path γ , $\gamma(0) = a$, $\gamma(1) = b$. The path is compact, because it is the image $\gamma([0, 1])$. The set of open sets $\{B(x, r_x) | x \in \gamma([0, 1])\}$ covers the path. Therefore there is a finite subcover $\{B(x_i, r_{x_i})\}$. One of these balls must contain a and therefore overlaps with U . Thus w is identically zero on this ball. Proceeding inductively, w is zero on every ball, and in particular it is zero at b . Since this holds for all $b \in \Omega$, $w \equiv 0$.

Notice similarity of this argument to the proof of the maximum principle.

35. To be or not to be.

Consider the Dirichlet problem for the Laplace equation $\Delta u = 0$ on Ω with $u = g$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^n$ is an open and bounded subset and g is a continuous function. We know from the weak maximum principle that there is at most one solution. In this question we see that for some domains, existence is not guaranteed.

- (a) Consider $\Omega = B(0, 1) \setminus \{0\}$, so that the boundary $\partial\Omega = \partial B(0, 1) \cup \{0\}$ consists of two components. We write $g(x) = g_1(x)$ for $x \in \partial B(0, 1)$ and $g(0) = g_2$. Show that there does not exist a solution for $g_1(x) = 0$ and $g_2 = 1$.

Hint. Use Lemma 3.24. (3 points)

- (b) Generalise this: What are the necessary and sufficient conditions on g for the Dirichlet problem to have a solution on this domain? (3 points)

- (c) Generalise again: What can you say about the Dirichlet problem for bounded domains whose boundaries have isolated points? (1 point)

Solution.

- (a) Suppose for contradiction that u exists. From the definition of the problem, we know that u is a continuous function on the ball, but we do not know that u is harmonic at 0; the

PDE $\Delta u = 0$ is only taken to hold on Ω . However, u is bounded on the ball, so Lemma 3.24 allows us to conclude that u is in fact harmonic on the ball $B(0, 1)$. Thus u obeys $\Delta u = 0$ on all of $B(0, 1)$. By the weak maximum principle, the only harmonic function on $B(0, 1)$ which is identically zero on $\partial B(0, 1)$ is $u \equiv 0$. That contradicts the boundary condition $u(0) = 1$.

- (b) From Poisson's Representation Formula 3.21 that there is a unique solution v to the Dirichlet problem of the Laplace equation on the ball $B(0, 1)$ with boundary condition $v|_{\partial B(0,1)} = g_1$. If this v has $v(0) = g_2$ then it is a solution to the Dirichlet problem on Ω . (Sufficient condition).

On the other hand, if $v(0) \neq g_2$ then there is no solution. For contradiction, suppose there were a solution u . Then $w = u - v$ is also harmonic on Ω and has $w|_{\partial B(0,1)} = g_2 - g_2 = 0$ but $w(0) \neq 0$. Applying the argument of part (a) to w gives a contradiction.

- (c) Let Ω be a bounded domain with an isolated point x_0 in its boundary. Then $\Omega' = \Omega \cup \{x_0\}$ is also an open bounded domain. By Lemma 3.24 any solution to the Dirichlet problem on Ω is a solution to the Dirichlet problem on Ω' . Therefore there is a solution on Ω if and only if there is a solution v on Ω' with $v(x_0) = g(x_0)$.

