Martin Schmidt
Ross Ogilvie

Introduction to Partial Differential Equations
Exercise sheet 9

1st November, 2022

This is a public holiday. Perhaps we will reschedule the tutorial. Either way, you must submit the exercise sheet.

32. Do nothing by halves.

Let $H_1^+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 > 0\}$ be the upper half-space and $H_1^0 = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = 0\}$ the dividing hyperplane. We call $R_1(x) = (-x_1, x_2, \ldots, x_n)$ reflection in the plane H^0.

(a) A reflection principle for harmonic functions. Let $u \in C^2(\overline{H_1^+})$ be a harmonic function that vanishes on H_1^0. Show that the function $v : \mathbb{R}^n \to \mathbb{R}$ defined through reflection

$$v(x) = \begin{cases} u(x) & \text{for } x_1 \geq 0 \\ -u(R_1(x)) & \text{for } x_1 < 0 \end{cases}$$

is harmonic. *(4 points)*

(b) Green's function for the half-space. Show that Green's function for H_1^+ is

$$G(x, y) = \Phi(x - y) - \Phi(R_1(x) - y).$$

(3 points)

(c) Green's function for the half-ball. Compute the Green's function for B^+. *(3 points)*
Hint. Make use of both the Green's function for the ball 3.20 and part (b).

33. It's not easy being green.

Suppose that Ω is a bounded domain. Prove that there is at most one Green's function on Ω. *(3 points)*

On the other hand, suppose that Ω has a Green's function G_Ω and that there exists a non-trivial solution to the Dirichlet problem

$$\Delta u = 0, \quad u|_{\partial\Omega} = 0.$$

Construct a second Green's function for Ω. *(2 points)*

34. One of a kind.

Let $\Omega \subseteq \mathbb{R}^n$ be an open and connected domain and $u, v \in C^2(\overline{\Omega})$ two harmonic functions. Suppose that there is an open subset $U \subset \Omega$ such that $u = v$ on U. Prove that $u = v$ on Ω using Corollary 3.22 (Harmonic functions are analytic). This is called the *unique continuation property* of harmonic functions. *(3 points)*

35. To be or not to be.

Consider the Dirichlet problem for the Laplace equation $\Delta u = 0$ on Ω with $u = g$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^n$ is an open and bounded subset and g is a continuous function. We know from the weak maximum principle that there is at most one solution. In this question we see that for some domains, existence is not guaranteed.

(a) Consider $\Omega = B(0,1) \setminus \{0\}$, so that the boundary $\partial\Omega = \partial B(0,1) \cup \{0\}$ consists of two components. We write $g(x) = g_1(x)$ for $x \in \partial B(0,1)$ and $g(0) = g_2$. Show that there does not exist a solution for $g_1(x) = 0$ and $g_2 = 1$.

Hint. Use Lemma 3.24. *(3 points)*

(b) Generalise this: What are the necessary and sufficient conditions on g for the Dirichlet problem to have a solution on this domain? *(3 points)*

(c) Generalise again: What can you say about the Dirichlet problem for bounded domains whose boundaries have isolated points? *(1 point)*