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Introduction to Partial Differential Equations
Exercise sheet 8

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28. Weak Tea.

In this question we try to generalise the idea of spherical means to distributions. Let $\psi \in C_0^\infty((0,\infty))$ and define

$$f_{x,\psi}(y) = \frac{1}{n\omega_n\,|y-x|^{n-1}}\psi(|y-x|)$$

as in the weak mean value property.

(a) Describe the support of $f_{x,\psi}$ in terms of the support of ψ. *(1 point)*

(b) Let λ_ε be a family of mollifiers on $\mathbb{R}$. State the properties of a family of mollifiers. *(1 point)*

(c) Set $\psi_{r,\varepsilon}(t) = \lambda_\varepsilon(t-r)$ for some $r > 0$. Suppose that g is a continuous function. Show that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} g\, f_{x,\psi_{r,\varepsilon}} = \mathcal{S}(g,x,r),$$

the spherical mean of g. *(3 points)*

Hint. Write this as an integral over a ball, and then as an integral over integrals of spheres.

(d) Because of this, we may try to define the spherical mean of a distribution F as $\lim_{\varepsilon\to 0} F(f_{x,\psi_{r,\varepsilon}})$. However this does not always exist.

Let G be the distribution in Exercise 19(d), integration on the unit circle. Show that $G(f_{0,\psi}) = \psi(1)$ for any appropriate ψ. Try to compute limit from the previous part with $r = 1$. *(2 points)*

(e) Show that the limit does exist for all harmonic distributions. *(2 points)*

Solution.

(a) The support of $f_{x,\psi}$ is the points y with $|y-x| \in \operatorname{supp}\psi$. Consider $|y-x|$ as a radius. Then y belongs to the support of f if and only if it belongs to $\partial B(x,r)$ and r belongs to the support of ψ. The support of f is a union of spheres centered at x.

(b) These should be non-negative functions. Their support should be contained in $[-\varepsilon,\varepsilon]$. Their integral over the real line should be 1.

(c) Since $\psi_{r,\varepsilon}$ has compact support which decreases as ε does, the support of f is contained in some large ball $B(x,M)$. As suggested by the hint, we write the integral over this ball as an iterated integral of spheres

$$\int_{\mathbb{R}^n} g\, f_{x,\psi_{r,\varepsilon}} = \int_{B(x,M)} g\, f_{x,\psi_{r,\varepsilon}} = \int_0^M \int_{\partial B(x,s)} g\, f_{x,\psi_{r,\varepsilon}}\, d\sigma\, ds.$$

Next recognise that f is in fact constant on the sphere $\partial B(x,s)$, ie $|y-x| = s$, so we can bring it outside the inner integral

$$= \int_0^M \psi_{r,\varepsilon}(s)\frac{1}{n\omega_n\, s^{n-1}}\int_{\partial B(x,r)} g\, d\sigma\, ds = \int_0^M \psi_{r,\varepsilon}(s)\mathcal{S}(g,x,s)\, ds.$$

Now we use the specific form of $\psi_{r,\varepsilon}$ and a change of coordinates

$$= \int_0^M \lambda_\varepsilon(s-r)\mathcal{S}(g,x,s)\,ds = \int_{-\infty}^{\infty} \lambda_\varepsilon(t)\mathcal{S}(g,x,t+r)\,dt.$$

Finally, we know that when we integrate a continuous function against a mollifier and take the limit as $\varepsilon \to 0$ we get the function at $t=0$ (Lemma 2.8). Thus

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} g\, f_{x,\psi_{r,\varepsilon}} = \mathcal{S}(g,x,0+r) = \mathcal{S}(g,x,r).$$

Note that this holds for all families of mollifiers, so the operation is well-defined.

(d) Let G be the distribution from Exercise 19(d). Then

$$G(f_{0,\psi}) = \int_C \frac{1}{n\omega_n\, 1^{n-1}}\psi(1)\,d\sigma = \psi(1).$$

So we consider $\psi(t) = \psi_{1,\varepsilon}(t) = \lambda_\varepsilon(t-1)$. This has support in $[1-\varepsilon, 1+\varepsilon]$ and the integral is 1 for all ε. So for a typical mollifier with $\lambda_\varepsilon(1-1) \to \infty$. Hence the limit does not exist.

(e) We have seen in the weak mean value property that a harmonic distribution vanishes on the test function $f_{x,\psi}$ when $\int \psi = 0$. Choose any test function χ on the positive axis with total integral 1. Then $\psi(t) = \chi(t) - \lambda_\varepsilon(t-1)$ is a test function with total integral zero. Notice that $f_{x,\psi} = f_{x,\chi} - f_{x,\lambda_\varepsilon}$ Hence

$$0 = U(f_{x,\psi}) = U(f_{x,\chi} - f_{x,\lambda_\varepsilon}) = U(f_{x,\chi}) - U(f_{x,\lambda_\varepsilon}).$$

In other words $U(f_{x,\lambda_\varepsilon})$ is constant as ε changes. Thus the limit exists and is equal to $U(f_{x,\chi})$, *for any χ with total integral one.* It is for this reason that we can use this idea of a spherical mean of a distribution in Weyl's lemma without needing to take $\varepsilon \to 0$ or even use mollifiers at all.

29. Back in the saddle.

Suppose that $u \in C^2(\mathbb{R}^2)$ is a harmonic function with a critical point at x_0. Assume that the Hessian of u has non-zero determinant. Show that x_0 is a saddle point. Explain the connection to the maximum principle. *(2 points)*

Solution. Let $a = \partial_x^2 u$, $b = \partial_x\partial_y u$, $c = \partial_y^2 u$ at x_0. Because u is harmonic, $a + d = 0$. The determinant of the Hessian of u at x_0 is $ad - b^2 = -a^2 - b^2 \leq 0$. Since by assumption it is not zero, it is negative. Therefore x_0 is a saddle point.

The maximum principle implies that any extrema must occur on the boundary. We know from Analysis I or II that extrema can only occur on the boundary or at critical points. It follows that harmonic functions can only have critical points that are saddle points.

30. Subharmonic Functions

Let $\Omega \subset \mathbb{R}^n$ be an open and connected region. A continuous function $v : \overline{\Omega} \to \mathbb{R}$ is called *subharmonic* if for all $x \in \Omega$ and $r > 0$ with $B(x,r) \subset \Omega$ it lies below its spherical mean: $v(x) \le \mathcal{S}(v, x, r)$.

(a) Prove that every subharmonic function obeys the *maximum principle*: If the maximum of v can be found inside Ω then v is constant. *(2 points)*

(b) Suppose that v is twice continuous differentiable. Show that v is subharmonic if and only if $-\Delta v \le 0$ in Ω. *(3 points)*

(c) Let $u : \overline{\Omega} \to \mathbb{R}$ be a harmonic function. Show that $\|\nabla u\|^2$ is subharmonic. *(2 points)*

(d) Show that $f \circ u$ is subharmonic for any smooth convex function $f : \mathbb{R} \to \mathbb{R}$. *(2 points)*

(e) Let v_1, v_2 be two subharmonic functions. Show that $v = \max(v_1, v_2)$ is subharmonic. *(1 point)*

Solution.

(a) Suppose that v does indeed obtain a maximum at some $x_0 \in \Omega$, so that $v(x_0) - v(x) \ge 0$. Let $B(x_0, R) \subset \Omega$. The mean of a constant is just the same constant $\mathcal{S}(v(x_0), y, r) = v(x_0)$. For $r < R$ we have that

$$0 \ge v(x_0) - \mathcal{S}(v, x_0, r) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x_0,r)} v(x_0) - v(x) \, \mathrm{d}\sigma(x) \ge 0.$$

It follows that $v(x_0) - v(x) = 0$ for all $x \in B(x_0, r)$ (we will show this below, if you are unsure how to prove this). It follows that $v(x) = v(x_0)$ for all points $x \in B(x_0, R)$.

Repeating this argument we can show that v is also the same constant on any ball that overlaps $B(x_0, R)$, and so on. Because Ω is connected, this shows that $v(x) = v(x_0)$ on all of Ω. (The formal version of this argument, using paths and finite covering by balls, is in the lecture script.)

Let us justify the claim made above. Suppose that f is continuous and non-negative, and

$$\int_{B(0,r)} f \, d\sigma = 0.$$

We want to show that $f(x) = 0$ for all $x \in B(0, r)$. Suppose there is a point x_1 where $f(x_1) > 0$. Then by continuity there is an open ball $U = B(x_1, \varepsilon)$ such that for all $x \in U$ it holds that $f(x) \ge \frac{1}{2} f(x_1)$. Then

$$\begin{aligned}\int_{\partial B(0,r)} f(x) \, \mathrm{d}\sigma(x) &\ge \int_{\partial B(0,r) \cap U} f(x) \, \mathrm{d}\sigma(x) \ge \int_{\partial B(0,r) \cap U} \frac{1}{2} f(x_1) \, \mathrm{d}\sigma(x) \\ &\ge \frac{1}{2} f(x_1) \, \sigma(\partial B(0,r) \cap U) \\ &> 0\end{aligned}$$

because $\partial B(x_0, r) \cap U$ is an open subset of the sphere and therefore has positive measure. This is a contradiction. Therefore such x_1 cannot exist; $f(x)$ is zero on the sphere.

A note on the need for this proof. Notice it is not enough to use a neighbourhood U where $f(x) > 0$ because integrals do not preserve *strict* inequalities. Hence we use this trick with $f(x) > \frac{1}{2}f(x_1)$ so in the last step we can apply a strict inequality. This is a common trick in analysis when dealing with continuous functions and limits (remember, the integral is a limit of a sum).

Indeed, if $f < g$ are integrable functions we can't say $\int_E f < \int_E g$. For example, take E to be a null set, so that both integrals are zero. However this is the only counterexample. Try to prove this yourself, and feel free to ask for a hint/help. The proof I give above though, which only applies to continuous functions, is meant to be more intuitive and geometric. It avoids using facts about the surface measure $d\sigma$ other than $\partial B(x_0, r) \cap U$ has positive measure.

(b) We know from a previous exercise that

$$\frac{\partial}{\partial r}\mathcal{S}(v, x, r) = \frac{1}{n\omega_n}\int_{B(0,1)} \Delta v(x_0 + rz)\, \mathrm{d}z.$$

Suppose that $\Delta v \geq 0$. Then this shows that $\mathcal{S}(v, x, r)$ is a non-decreasing function. On the other hand $v(x) = \mathcal{S}(v, x, 0)$. Thus $v(x) = \mathcal{S}(v, x, 0) \leq \mathcal{S}(v, x, r)$.

For the converse, we can not say from $v(x) = \mathcal{S}(v, x, 0) \leq \mathcal{S}(v, x, r)$ directly that $\mathcal{S}$ is a non-decreasing function of r. For example, $\mathcal{S}$ could initially increase, but then oscillate. But we do have the intuition that it must increase initially. We use the following argument to prove this rigorously.

Suppose that $v(x) \leq \mathcal{S}(v, x, r)$ for all valid x, r but also that there is a point x_0 with $\Delta v(x_0) < 0$. By continuity there is a ball $B(x_0, R)$ on which $\Delta v(x) < \frac{1}{2}\Delta v(x_0)$. But then for all $r < R$

$$\frac{\partial}{\partial r}\mathcal{S}(v, x_0, r) = \frac{1}{n\omega_n}\int_{B(0,1)} \Delta v(x_0 + rz)\, \mathrm{d}z \leq \frac{1}{n\omega_n}\int_{B(0,1)} \frac{1}{2}\Delta v(x_0)\, \mathrm{d}z = \frac{1}{2}\Delta v(x_0).$$

Integrating this from $r = 0$ to R gives

$$\mathcal{S}(v, x_0, R) \leq \frac{1}{2}\Delta v(x_0)R + \mathcal{S}(v, x_0, 0) = \frac{1}{2}\Delta v(x_0)R + v(x_0) < v(x_0).$$

This is similar to the proof in (a), in that we use the strict estimate $\frac{1}{2}\Delta v(x_0) < 0$ in the final step. This shows that $v(x_0)$ lies above the spherical mean $\mathcal{S}(v, x_0, R)$ and so is not subharmonic. (Actually, it lies above all the spherical means $\mathcal{S}(v, x_0, r)$ for $r \leq R$.) The contrapositive statement is that if v is subharmonic then there are no points with $\Delta v < 0$.

(c) This follows by direct computation

$$\begin{aligned}\Delta\|\nabla u\|^2 &= \sum_{j,k=1}^{n} \partial_j^2\left[(\partial_k u)^2\right] = \sum_{j,k=1}^{n} \partial_j\left[2\partial_k u\, \partial_j\partial_k u\right] = 2\sum_{j,k=1}^{n} \partial_j\partial_k u\, \partial_j\partial_k u + \partial_k u\, \partial_j^2\partial_k u \\ &= 2\sum_{j,k=1}^{n} (\partial_j\partial_k u)^2 + 2\sum_{k=1}^{n} \partial_k u\, \partial_k\Delta u = 2\sum_{j,k=1}^{n} (\partial_j\partial_k u)^2 \geq 0.\end{aligned}$$

There is a nice proof using the next part:

(i) $\partial_k u$ is harmonic, by previous exercise.

(ii) $(\partial_k u)^2$ is subharmonic, since $f(u) = u^2$ is convex.

(iii) $\|\nabla u\|^2 = \sum(\partial_k u)^2$ is subharmonic, because it is the sum of subharmonic functions.

(d) A smooth convex function has the property that $f'' \geq 0$. By the chain rule then

$$\begin{aligned}\Delta(f \circ u) &= \sum_{j=1}^{n} \partial_j \partial_j (f \circ u) = \sum_{j=1}^{n} \partial_j (f' \circ u) \partial_j u = \sum_{j=1}^{n} (f'' \circ u)(\partial_j u)^2 + (f' \circ u) \partial_j^2 u \\ &= (f'' \circ u) \|\nabla u\|^2 + (f' \circ u) \Delta u \geq 0\end{aligned}$$

(e) The inequality $v_i(x) \leq v(x)$ for all points gives

$$\frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} v_i(y) \, \mathrm{d}\sigma(y) \leq \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} v(y) \, \mathrm{d}\sigma(y)$$

and so $v_i(x) \leq \mathcal{S}(v_i, x, r) \leq \mathcal{S}(v, x, r)$. It follows $v(x) \leq \mathcal{S}(v, x, r)$.

31. Never judge a book by its cover.

Let $\Omega \subset \mathbb{R}^n$ be an open, connected, and bounded subset, and let $f : \Omega \to \mathbb{R}$ and $g_1, g_2 : \partial\Omega \to \mathbb{R}$ be continuous functions. Consider then the two Dirichlet problems

$$-\Delta u = f \text{ on } \Omega, \qquad u|_{\partial\Omega} = g_k,$$

for $k = 1, 2$. Let u_1, u_2 be respective solutions such that they are twice continuously differentiable on Ω and continuous on $\overline{\Omega}$. Show that if $g_1 \leq g_2$ on $\partial\Omega$ then $u_1 \leq u_2$ on Ω. *(4 points)*

Solution. Let $v = u_2 - u_1$. This is a harmonic function and on the boundary it is equal to $g_2 - g_1 \geq 0$. If v is negative at any point, then this would contradict the maximum principle by being lower than the boundary. Hence $v \geq 0$, which is exactly that $u_1 \leq u_2$.