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Introduction to Partial Differential Equations
Exercise sheet 8

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28. Weak Tea.

In this question we try to generalise the idea of spherical means to distributions. Let $\psi \in C_0^\infty((0,\infty))$ and define

$$f_{x,\psi}(y) = \frac{1}{n\omega_n \, |y-x|^{n-1}} \psi(|y-x|)$$

as in the weak mean value property.

(a) Describe the support of $f_{x,\psi}$ in terms of the support of ψ. *(1 point)*

(b) Let λ_ε be a family of mollifiers on $\mathbb{R}$. State the properties of a family of mollifiers. *(1 point)*

(c) Set $\psi_{r,\varepsilon}(t) = \lambda_\varepsilon(t-r)$ for some $r > 0$. Suppose that g is a continuous function. Show that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} g \, f_{x,\psi_{r,\varepsilon}} = \mathcal{S}(g,x,r),$$

the spherical mean of g. *(3 points)*

Hint. Write this as an integral over a ball, and then as an integral over integrals of spheres.

(d) Because of this, we may try to define the spherical mean of a distribution F as $\lim_{\varepsilon\to 0} F(f_{x,\psi_{r,\varepsilon}})$. However this does not always exist.

Let G be the distribution in Exercise 19(d), integration on the unit circle. Show that $G(f_{0,\psi}) = \psi(1)$ for any appropriate ψ. Try to compute limit from the previous part with $r = 1$. *(2 points)*

(e) Show that the limit does exist for all harmonic distributions. *(2 points)*

29. Back in the saddle.

Suppose that $u \in C^2(\mathbb{R}^2)$ is a harmonic function with a critical point at x_0. Assume that the Hessian of u has non-zero determinant. Show that x_0 is a saddle point. Explain the connection to the maximum principle. *(2 points)*

30. Subharmonic Functions

Let $\Omega \subset \mathbb{R}^n$ be an open and connected region. A continuous function $v : \overline{\Omega} \to \mathbb{R}$ is called *subharmonic* if for all $x \in \Omega$ and $r > 0$ with $B(x,r) \subset \Omega$ it lies below its spherical mean: $v(x) \le \mathcal{S}(v,x,r)$.

(a) Prove that every subharmonic function obeys the *maximum principle*: If the maximum of v can be found inside Ω then v is constant. *(2 points)*

(b) Suppose that v is twice continuous differentiable. Show that v is subharmonic if and only if $-\Delta v \le 0$ in Ω. *(3 points)*

(c) Let $u : \overline{\Omega} \to \mathbb{R}$ be a harmonic function. Show that $\|\nabla u\|^2$ is subharmonic. *(2 points)*

(d) Show that $f \circ u$ is subharmonic for any smooth convex function $f : \mathbb{R} \to \mathbb{R}$. *(2 points)*

(e) Let v_1, v_2 be two subharmonic functions. Show that $v = \max(v_1, v_2)$ is subharmonic.

(1 point)

31. Never judge a book by its cover.

Let $\Omega \subset \mathbb{R}^n$ be an open, connected, and bounded subset, and let $f : \Omega \to \mathbb{R}$ and $g_1, g_2 : \partial\Omega \to \mathbb{R}$ be continuous functions. Consider then the two Dirichlet problems

$$-\Delta u = f \text{ on } \Omega, \qquad u|_{\partial\Omega} = g_k,$$

for $k = 1, 2$. Let u_1, u_2 be respective solutions such that they are twice continuously differentiable on Ω and continuous on $\overline{\Omega}$. Show that if $g_1 \leq g_2$ on $\partial\Omega$ then $u_1 \leq u_2$ on Ω. *(4 points)*