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Introduction to Partial Differential Equations
Exercise sheet 7

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23. Twirling towards freedom.

Let $u \in C^2(\mathbb{R}^n)$ be a harmonic function. Show that the following functions are also harmonic.

(a) $v(x) = u(x+b)$ for $b \in \mathbb{R}^n$.

(b) $v(x) = u(ax)$ for $a \in \mathbb{R}$.

(c) $v(x) = u(Rx)$ for $R(x_1, \dots, x_n) = (-x_1, x_2, \dots, x_n)$ the reflection operator.

(d) $v(x) = u(Ax)$ for any orthogonal matrix $A \in O(\mathbb{R}^n)$.

Together these show that the Laplacian is invariant under all Euclidean motions and harmonic functions can be rescaled. *(6 points)*

Solution.

(a) This follows by the chain rule

$$\Delta v(x) = \sum \frac{\partial^2 u}{\partial x_i^2}(x+b) \cdot 1 = \Delta u(x+b) = 0.$$

(b) This also follows by the chain rule

$$\Delta v(x) = \sum \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}(ax) \cdot a\right) = a^2 \Delta u(ax) = 0.$$

(c) You guessed it, we apply the chain rule. Only the x_1 derivative is affected:

$$\frac{\partial^2 v}{\partial x_1^2}(x) = \frac{\partial}{\partial x_1}\left(-\frac{\partial u}{\partial x_1}(Rx)\right) = \frac{\partial^2 u}{\partial x_1^2}(Rx).$$

This shows $\Delta v(x) = \Delta u(Rx) = 0$.

(d) This is also the chain rule, with $(Ax)_i = \sum_j A_{ij} x_j$. We will write this with indices, but if you can keep everything as matrices then it is a bit shorter.

$$\begin{aligned}
\frac{\partial v}{\partial x_k}(x) &= \frac{\partial}{\partial x_k} u\Big(\sum_j A_{ij} x_j\Big) \\
&= \sum_l \frac{\partial u}{\partial x_l}\Big(\sum_j A_{ij} x_j\Big) A_{lk} \\
\frac{\partial^2 v}{\partial x_k^2}(x) &= \frac{\partial}{\partial x_k} \sum_l \frac{\partial u}{\partial x_l}\Big(\sum_j A_{ij} x_j\Big) A_{lk} \\
&= \sum_{l,m} \frac{\partial^2 u}{\partial x_m \partial x_l}\Big(\sum_j A_{ij} x_j\Big) A_{lk} A_{mk}
\end{aligned}$$

Now when we sum over k, we can group together the like derivatives and get a sum over the A multipliers. Because A is an orthogonal matrix, we have $AA^T = I$, or in other words $\delta_{lm} = \sum_k A_{lk}(A^T)_{km} = \sum_k A_{lk} A_{mk}$. This gives

$$\Delta v(x) = \sum_{l,m} \frac{\partial^2 u}{\partial x_m \partial x_l}(Ax)\left(\sum_k A_{lk} A_{mk}\right) = \sum_{l,m} \frac{\partial^2 u}{\partial x_m \partial x_l}(Ax)\delta_{lm} = \sum_l \frac{\partial^2 u}{\partial x_l^2}(Ax) = 0.$$

24. Harmonic Polynomials in Two Variables.

(a) Let $u \in C^\infty(\mathbb{R}^n)$ be a smooth harmonic function. Prove that any derivative of u is also harmonic. *(1 point)*

(b) Choose any positive degree n. Consider the complex valued function $f_n : \mathbb{R}^2 \to \mathbb{C}$ given by $f_n(x, y) = (x+\iota y)^n$ and let $u_n(x, y)$ and $v_n(x, y)$ be its real and imaginary parts respectively. Show that u_n and v_n are harmonic. *(2 points)*

(c) A *homogeneous polynomial* of degree n in two variables is a polynomial of the form $p = \sum a_k x^k y^{n-k}$. Show that $\partial_x p$ and $\partial_y p$ are homogeneous of degree $n-1$. *(1 point)*

(d) Show that such a homogeneous polynomial of degree n is harmonic if and only if it is a linear combination of u_n and v_n. *(3 bonus points)*

Solution.

(a) Since the function is smooth, it is in particular thrice continuously differentiable. Thus we can interchange the order of partial derivatives

$$\Delta(\partial_i u) = \sum_k \partial_k^2 \partial_i u = \sum_k \partial_i \partial_k^2 u = \partial_i \Delta u = 0.$$

(b) There are two approaches. The simplest is to extend the Laplacian linearly to complex valued functions. All normal rules of calculus apply and we get

$$\Delta f_n = n(n-1)(x+\iota y)^{n-2} + n(n-1)(\iota^2)(x+\iota y)^{n-2} = 0.$$

But perhaps this feels undeserved. Let's instead compute more directly. By binomial expansion we have

$$u_n + \iota v_n = \sum_{k=0}^{n} \binom{n}{k} \iota^k x^{n-k} y^k = \sum_{0 \le 2j \le n} \binom{n}{2j} (-1)^j x^{n-2j} y^{2j} + \iota \sum_{0 \le 2j+1 \le n} \binom{n}{2j+1} (-1)^{2j} x^{n-2j-1} y^{2j+1}.$$

Differentiating gives

$$\begin{aligned}\Delta u_n &= \sum_{0 \le 2j \le n-1} (n-2j)(n-2j-1)\binom{n}{2j}(-1)^j x^{n-2j-2} y^{2j} + \sum_{1 \le 2j \le n} (2j)(2j-1)\binom{n}{2j}(-1)^j x^{n-2j} y^{2j-2} \\ &= \sum_{0 \le 2j \le n-1} \left[(n-2j)(n-2j-1)\binom{n}{2j} - (2j+2)(2j+1)\binom{n}{2j+2} \right] (-1)^j x^{n-2j-2} y^{2j}.\end{aligned}$$

The result now follows from the definition of the binomial coefficients.

$$\begin{aligned}(n-2j)(n-2j-1)\binom{n}{2j} &= (n-2j)(n-2j-1)\frac{n!}{(n-2j)!(2j)!} = \frac{n!}{(n-2j-2)!(2j)!} \\ (2j+2)(2j+1)\binom{n}{2j+2} &= (2j+2)(2j+1)\frac{n!}{(n-2j-2)!(2j+2)!} = \frac{n!}{(n-2j-2)!(2j)!}\end{aligned}$$

Likewise for v_n.

(c)

$$\partial_x \sum_{k=0}^{n} a_k x^k y^{n-k} = \sum_{k=1}^{n} k a_k x^{k-1} y^{n-k} = \sum_{j=0}^{n-1} (j+1) a_{j+1} x^j y^{(n-1)-j}$$

(d) Any linear combination of u_n and v_n is harmonic since Δ is a linear operator. For the converse, we prove this by induction.

For $n = 0$ and $n = 1$ the result holds because u_n, v_n span all polynomials.

Suppose now it holds up to degree n. Let $p = p_0 x^{n+1} y^0 + p_1 x^n y^1 + \dots$ be a homogeneous harmonic polynomial of degree $n+1$. Define $q = p - p_0 u_{n+1} - \frac{1}{n} p_1 v_{n+1}$. Then this does not have the terms $x^{n+1}y^0$ or $x^n y^1$. Note that $\partial_x q$ is again a homogeneous harmonic polynomial and its degree is n, so $\partial_x q = a u_n + b v_n$ for some constants a and b. But $\partial_x q$ has no term with $x^n y^0$ or $x^{n-1} y^1$, hence $a = b = 0$. This shows that q is constant with respect to x, and the only possibility is then that $q = A y^{n+1}$. But this is only harmonic for $A = 0$. We conclude therefore that $p = p_0 u_{n+1} + \frac{1}{n} p_1 v_{n+1}$.

25. Means and Ends

In the lecture script we often encounter the *spherical mean* of a function $f : \Omega \to \mathbb{R}$:

$$\mathcal{S}(f, x, r) := \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} f(y)\, \mathrm{d}\sigma(y).$$

If x is in the interior of Ω, then there exists a ball $B(x, R) \subset \Omega$. The spherical mean is then defined for all $0 < r < R$.

Suppose that f is continuous. Prove $\lim_{r \to 0^+} \mathcal{S}(f, x, r) = f(x)$. *(4 points)*

Let $f \in C^2(\overline{\Omega})$ be any twice continuously differentiable function. Carefully justify the formula

$$\frac{\partial}{\partial r} \mathcal{S}(f, x, r) = \frac{1}{n\omega_n} \int_{B(0,1)} \Delta f(x + rz)\, \mathrm{d}z.$$

This formula is used in the proof of the Mean Value property. It shows why spherical means and harmonic functions are related. *(5 points)*

Solution. We first prove some useful properties of the spherical mean. The spherical mean of a constant is the same constant $\mathcal{S}(c, x, r) = c$, because of the normalising factor in front of the integral. It also follows from the property of the integral that $\mathcal{S}$ is $\mathbb{R}$-linear in the function $\mathcal{S}(af + bg, x, r) = a\mathcal{S}(f, x, r) + b\mathcal{S}(g, x, r)$. Finally, it is order preserving: if $f \leq g$ for all points then $\mathcal{S}(f, x, r) \leq \mathcal{S}(g, x, r)$.

Now, this is a single variable limit, the other variables are being held constant. So

$$\begin{aligned}|\mathcal{S}(f,x,r)-f(x)| = |\mathcal{S}(f,x,r)-\mathcal{S}(f(x),x,r)| &= |\mathcal{S}(f-f(x),x,r)| \leq \mathcal{S}(|f-f(x)|,x,r)\\ &= \frac{1}{n\omega_n r^{n-1}}\int_{\partial B(x,r)} |f(y)-f(x)| \ \mathrm{d}\sigma(y)\\ &\leq \frac{1}{n\omega_n r^{n-1}}\int_{\partial B(x,r)} \sup_{z\in\partial B(x,r)} |f(z)-f(x)| \ \mathrm{d}\sigma(y)\\ &= \sup_{z\in\partial B(x,r)} |f(z)-f(x)|\,.\end{aligned}$$

This upper bound is also a function of r and by the continuity of f it converges to 0 as $r \to 0$. Therefore $\mathcal{S}(f,x,r)$ converges to $f(x)$ as required.

First consider how the change of coordinates $y = G(z) = x + rz$ changes a parameter integral. If $\phi : \Omega \to \mathbb{R}^n$ is a parametrisation of the unit sphere $\partial B(0,1)$ then $\tilde{\phi}(t) = G\circ\phi(t) = x + r\phi(t)$ is a parametrisation of the sphere $\partial B(x,r)$. Note that

$$\det \tilde{\phi}'^T\tilde{\phi}' = \det r^2\phi'^T\phi' = r^{2(n-1)}\det\phi'^T\phi'$$

so according to the definition of parameter integrals

$$\int\limits_{\partial B(x,r)} f\, d\sigma = \int_\Omega v\circ\tilde{\phi}\sqrt{\det\tilde{\phi}'^T\tilde{\phi}'}\, d\sigma = r^{n-1}\int_\Omega f\circ G\circ\phi\sqrt{\det\phi'^T\phi'}\, d\sigma = r^{n-1}\int\limits_{\partial B(0,1)} f\circ G\, d\sigma$$

This explains the following line from the proof of Mean Value Property 3.3:

$$\mathcal{S}(v,x,r) := \frac{1}{n\omega_n r^{n-1}}\int_{\partial B(x,r)} f(y)\,\mathrm{d}\sigma(y) = \frac{1}{n\omega_n}\int_{\partial B(0,1)} f(x+rz)\,\mathrm{d}\sigma(z),$$

which removes the dependence of surface from r and also cancels the $r^{-(n-1)}$ factor. In fact $\mathcal{S}$ is well defined at $r = 0$, providing another proof of its continuity. For these reason it is common and convenient to define $\mathcal{S}(f,x,0) = f(x)$. Differentiating,

$$\frac{\partial}{\partial r}\mathcal{S}(f,x,r) = \frac{1}{n\omega_n}\int_{\partial B(0,1)}\frac{\partial}{\partial r}\Big(f(x_0+rz)\Big)\,\mathrm{d}\sigma(z).$$

The next step is to write this in the form of the divergence theorem. The normal of the unit ball is just z. Notice from the chain rule that

$$\frac{\partial}{\partial r}\Big(f(x_0+rz)\Big) = \partial_1 f\cdot z_1 + \cdots + \partial_n f\cdot z_n = \nabla f\cdot z.$$

Finally then we can apply the divergence theorem

$$\begin{aligned}\frac{\partial}{\partial r}\mathcal{S}(f,x,r) &= \frac{1}{n\omega_n}\int_{\partial B(0,1)} \nabla f(x_0+rz)\cdot N\,\mathrm{d}\sigma(z)\\ &= \frac{1}{n\omega_n}\int_{B(0,1)} \nabla\cdot\nabla f(x_0+rz)\,\mathrm{d}z = \frac{1}{n\omega_n}\int_{B(0,1)} \Delta f(x_0+rz)\,\mathrm{d}z.\end{aligned}$$

26. Liouville's Theorem.

Let $u \in C^2(\mathbb{R}^2)$ be a harmonic function. Liouville's theorem (3.5 in the script) says that if u is bounded, then u is constant. In this question we give a geometric proof using *ball means*. Similar to a spherical mean, the ball mean of a function $v \in C(\overline{\Omega})$ is defined when $\overline{B(x,r)} \subset \Omega$:

$$\mathcal{M}(v, x, r) = \frac{1}{\omega_n r^n} \int_{B(x,r)} v(y) \, \mathrm{d}y$$

This proof comes from the following article Nelson, 1961.

(a) Show that u obeys the mean value property on balls, $u(x) = \mathcal{M}(u, x, r)$.
(Hint. use a previous exercise to write the integral on the ball as an integral over the radius and the spheres.) *(2 points)*

(b) Consider two points a, b in the plane which are distance $2d$ apart. Now consider two balls, both with radius $r > d$, centred on the two points respectively. Show that the area of the intersection is *(2 bonus points)*

$$\operatorname{area} B(a,r) \cap B(b,r) = 2r^2 \operatorname{acos}(dr^{-1}) - 2d\sqrt{r^2 - d^2}$$

(c) Suppose that u is bounded on the plane: $-C \leq u(x) \leq C$ for all x and some constant C. Show that *(2 points)*

$$\left|\mathcal{M}(u, a, r) - \mathcal{M}(u, b, r)\right| \leq \frac{2C}{\omega_2}\left(\pi - 2\operatorname{acos}(dr^{-1}) - \frac{2d}{r}\sqrt{1 - d^2 r^{-2}}\right)$$

(d) Complete the proof that u is constant. *(2 points)*

Solution.

(a) We compute directly

$$\begin{aligned}
\mathcal{M}(u, x, r) &= \frac{1}{\omega_n r^n} \int_{B(x,r)} u(y) \, \mathrm{d}y = \frac{1}{\omega_n r^n} \int_0^r \int_{\partial B(x,\rho)} u(y) \, \mathrm{d}\sigma(y) \, \mathrm{d}\rho \\
&= \frac{1}{\omega_n r^n} \int_0^r n\omega_n \rho^{n-1} \mathcal{S}(u, x, \rho) \, \mathrm{d}\rho \\
&= \frac{n}{r^n} \int_0^r \rho^{n-1} u(x) \, \mathrm{d}\rho = \frac{n}{r^n} u(x) \int_0^r \rho^{n-1} \, \mathrm{d}\rho = u(x)
\end{aligned}$$

An alternate proof works by instead noting that we can replace the integral of u over the ball with the integral of $u(x)$ over the ball by the mean value property, and thereby avoid any integration:

$$\mathcal{M}(u, x, r) = \frac{1}{\omega_n r^n} \int_0^r \int_{\partial B(x,\rho)} u(y) \, \mathrm{d}\sigma(y) \, \mathrm{d}\rho = \frac{1}{\omega_n r^n} \int_0^r \int_{\partial B(x,\rho)} u(x) \, \mathrm{d}\sigma(y) \, \mathrm{d}\rho = u(x)$$

(b) Let the distance from the point where the two circles intersect to the line connecting a and b be h. We have then that $r^2 = d^2 + h^2$. It has an angle of elevation from the centre of the balls of $\cos\theta = d/r$. The area of the sector centred at b with angle 2θ is $\frac{1}{2} r^2 (2\theta) = r^2 \operatorname{acos}(dr^{-1})$. Subtracting off a triangle gives the area of the segment of $B(b,r)$ as

$$r^2 \operatorname{acos}(dr^{-1}) - dh.$$

Twice this is the area of $B(a,r) \cap B(b,r)$

(c) The integrals have the same integrand, so their difference reduces to integrals on certain domains:

$$\omega_2 r^2 \big(\mathcal{M}(u,a,r) - \mathcal{M}(u,b,r)\big) = \int_{B(a,r)\setminus B(b,r)} v(y)\ \mathrm{d}y - \int_{B(b,r)\setminus B(a,r)} v(y)\ \mathrm{d}y.$$

With the triangle inequalities

$$\begin{aligned}
\omega_2 r^2 \big|\mathcal{M}(u,a,r) - \mathcal{M}(u,b,r)\big| &\le \left| \int_{B(a,r)\setminus B(b,r)} u(y)\ \mathrm{d}y \right| + \left| \int_{B(b,r)\setminus B(a,r)} u(y)\ \mathrm{d}y \right| \\
&\le \int_{B(a,r)\setminus B(b,r)} |u(y)|\ \mathrm{d}y + \int_{B(b,r)\setminus B(a,r)} |u(y)|\ \mathrm{d}y \\
&\le C \int_{B(a,r)\setminus B(b,r)} \mathrm{d}y + C \int_{B(b,r)\setminus B(a,r)} \mathrm{d}y \\
&= 2C \left(\pi r^2 - \operatorname{area} B(a,r) \cap B(b,r)\right)
\end{aligned}$$

(d) By part (a), we know that a harmonic function is equal to its ball mean

$$|u(a) - u(b)| = \big|\mathcal{M}(u,a,r) - \mathcal{M}(u,b,r)\big| \le \frac{2C}{\omega_2} \left(\pi - 2\mathrm{acos}(dr^{-1}) - \frac{2d}{r}\sqrt{1 - d^2 r^{-2}} \right).$$

This must hold for all $r > 1$. But as $r \to \infty$ the right hand side tends to 0. By the squeeze rule, it must be that $u(a) = u(b)$. Since this holds for all pairs of points, u is constant.

Note: As stated in the linked article, this proof holds in all dimensions. The key trick to bound from below the volume of the lens (the overlap between the two balls) by the volume of a ball of radius $r - d$. This avoids having to calculate the volume of the lens exactly.