Martin Schmidt
Ross Ogilvie

Introduction to Partial Differential Equations
Exercise sheet 7

18th October, 2022

23. Twirling towards freedom.

Let $u \in C^2(\mathbb{R}^n)$ be a harmonic function. Show that the following functions are also harmonic.

(a) $v(x) = u(x + b)$ for $b \in \mathbb{R}^n$.

(b) $v(x) = u(ax)$ for $a \in \mathbb{R}$.

(c) $v(x) = u(Rx)$ for $R(x_1, \dots, x_n) = (-x_1, x_2, \dots, x_n)$ the reflection operator.

(d) $v(x) = u(Ax)$ for any orthogonal matrix $A \in O(\mathbb{R}^n)$.

Together these show that the Laplacian is invariant under all Euclidean motions and harmonic functions can be rescaled. *(6 points)*

24. Harmonic Polynomials in Two Variables.

(a) Let $u \in C^\infty(\mathbb{R}^n)$ be a smooth harmonic function. Prove that any derivative of u is also harmonic. *(1 point)*

(b) Choose any positive degree n. Consider the complex valued function $f_n : \mathbb{R}^2 \to \mathbb{C}$ given by $f_n(x, y) = (x + \iota y)^n$ and let $u_n(x, y)$ and $v_n(x, y)$ be its real and imaginary parts respectively. Show that u_n and v_n are harmonic. *(2 points)*

(c) A *homogeneous polynomial* of degree n in two variables is a polynomial of the form $p = \sum a_k x^k y^{n-k}$. Show that $\partial_x p$ and $\partial_y p$ are homogeneous of degree $n - 1$. *(1 point)*

(d) Show that such a homogeneous polynomial of degree n is harmonic if and only if it is a linear combination of u_n and v_n. *(3 bonus points)*

25. Means and Ends

In the lecture script we often encounter the *spherical mean* of a function $f : \Omega \to \mathbb{R}$:

$$\mathcal{S}(f, x, r) := \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} f(y) \, \mathrm{d}\sigma(y).$$

If x is in the interior of Ω, then there exists a ball $B(x, R) \subset \Omega$. The spherical mean is then defined for all $0 < r < R$.

Suppose that f is continuous. Prove $\lim_{r \to 0^+} \mathcal{S}(f, x, r) = f(x)$. *(4 points)*

Let $f \in C^2(\overline{\Omega})$ be any twice continuously differentiable function. Carefully justify the formula

$$\frac{\partial}{\partial r} \mathcal{S}(f, x, r) = \frac{1}{n\omega_n} \int_{B(0,1)} \Delta f(x + rz) \, \mathrm{d}z.$$

This formula is used in the proof of the Mean Value property. It shows why spherical means and harmonic functions are related. *(5 points)*

26. Liouville's Theorem.

Let $u \in C^2(\mathbb{R}^2)$ be a harmonic function. Liouville's theorem (3.5 in the script) says that if u is bounded, then u is constant. In this question we give a geometric proof using *ball means*. Similar to a spherical mean, the ball mean of a function $v \in C(\overline{\Omega})$ is defined when $\overline{B(x,r)} \subset \Omega$:

$$\mathcal{M}(v, x, r) = \frac{1}{\omega_n r^n} \int_{B(x,r)} v(y)\, \mathrm{d}y$$

This proof comes from the following article Nelson, 1961.

(a) Show that u obeys the mean value property on balls, $u(x) = \mathcal{M}(u, x, r)$.
(Hint. use a previous exercise to write the integral on the ball as an integral over the radius and the spheres.) *(2 points)*

(b) Consider two points a, b in the plane which are distance $2d$ apart. Now consider two balls, both with radius $r > d$, centred on the two points respectively. Show that the area of the intersection is *(2 bonus points)*

$$\operatorname{area} B(a,r) \cap B(b,r) = 2r^2 \operatorname{acos}(dr^{-1}) - 2d\sqrt{r^2 - d^2}$$

(c) Suppose that u is bounded on the plane: $-C \le u(x) \le C$ for all x and some constant C. Show that *(2 points)*

$$\left|\mathcal{M}(u, a, r) - \mathcal{M}(u, b, r)\right| \le \frac{2C}{\omega_2}\left(\pi - 2\operatorname{acos}(dr^{-1}) - \frac{2d}{r}\sqrt{1 - d^2 r^{-2}}\right)$$

(d) Complete the proof that u is constant. *(2 points)*