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Introduction to Partial Differential Equations
Exercise sheet 6

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19. Distributions.

(a) Choose any compact set $K \subset \mathbb{R}$. Since it is bounded, there exists $R > 0$ with $K \subseteq [-R, R]$. Now choose any test function $\phi \in C_0^\infty(\mathbb{R})$ with compact support in K. Since it is continuous, $\sup_{x\in K} |\phi(x)|$ is finite. Prove the following inequality

$$\left| \int_{-\infty}^{\infty} |x|\phi(x)\, \mathrm{d}x \right| \leq R^2 \sup_{x\in K} |\phi(x)|$$

(2 points)

(b) Show directly from Definition 2.6 that the distribution associated to the absolute value function

$$A : C_0^\infty(\mathbb{R}) \to \mathbb{R},\ \phi \mapsto \int_{-\infty}^{\infty} |x|\phi(x)\, \mathrm{d}x$$

is in fact a distribution on $\mathbb{R}$. *(1 point)*

(c) Calculate and describe the first and second derivatives of A as a distribution.

(3 points)

(d) Consider the circle $C = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$. Show that

$$G(\varphi) := \int_C \varphi\, d\sigma$$

defines a distribution in $\mathcal{D}'(\mathbb{R}^2)$. Note that the $d\sigma$ indicates this is an integration over the submanifold C. Does there exist a locally integrable function $g : \mathbb{R}^2 \to \mathbb{R}$ with

$$G(\varphi) = \int_{\mathbb{R}^2} g\, \varphi\, \mathrm{d}x$$

for all $\varphi \in C_0^\infty(\mathbb{R})$? (Hint. Use Lemma 2.9) *(2 Points + 2 Bonus Points)*

Solution.

(a)

$$\begin{aligned}
\left| \int_{-\infty}^{\infty} |x|\phi(x)\, \mathrm{d}x \right| &\leq \int_{-\infty}^{\infty} |x||\phi(x)|\, \mathrm{d}x \\
&= \int_K |x||\phi(x)|\, \mathrm{d}x && \text{since } \phi \text{ is zero outside of } K \\
&\leq \int_{[-R,R]} |x||\phi(x)|\, \mathrm{d}x && \text{since the integrand is positive and } K \subset [-R, R] \\
&\leq \int_{[-R,R]} R \sup_{x\in K} |\phi(x)|\, \mathrm{d}x \\
&= R^2 \sup_{x\in K} |\phi(x)|.
\end{aligned}$$

This is a very common idea for integrals of test functions. Because their support is compact, it has finite area. And because the functions are continuous, they obtain a maximum. These two factors then bound the integral of the test function.

(b) For the reason we just articulated, test functions are always L^1. Therefore the integral is well-defined and finite. The integral is a linear operator. So the only property we need to show is that A is continuous with respect to the semi-norms. Choose any compact set K. Let ϕ be a test function supported on K. Then by part (a)

$$|A(\phi)| = \left| \int_{-\infty}^{\infty} |x| \phi(x) \, \mathrm{d}x \right| \le 2R^2 \, \|\phi\|_{K,0}.$$

Thus the required inequality holds with $M = 1$, $\alpha_1 = 0$, and $C_1 = 2R^2$.

(c) In one sense, calculating the derivatives are easy, they are just $\partial_x A(\phi) = -A(\partial_x \phi)$ and $\partial_x^2 A(\phi) = -\partial_x A(\partial_x \phi) = A(\partial_x^2 \phi)$. But this does not give us an insight into their behaviour. However

$$\begin{aligned}
\partial_x A(\phi) = -A(\partial_x \phi) = -\int_{-\infty}^{\infty} |x| \phi' \, \mathrm{d}x &= \int_{-\infty}^{0} x\phi' \, \mathrm{d}x - \int_{0}^{\infty} x\phi' \, \mathrm{d}x \\
&= \Big[x\phi \Big]_{-\infty}^{0} - \int_{-\infty}^{0} \phi \, \mathrm{d}x - \Big[x\phi \Big]_{0}^{\infty} + \int_{0}^{\infty} \phi \, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \Big(-\chi_{[-\infty,0]} + \chi_{[0,\infty]} \Big) \phi \, \mathrm{d}x.
\end{aligned}$$

Thus we see it's distribution is associated to the function $-\chi_{[-\infty,0]} + \chi_{[0,\infty]}$.

$$\begin{aligned}
\partial_x^2 A(\phi) = -\partial_x A(\partial_x \phi) &= \int_{-\infty}^{\infty} \Big(-\chi_{[-\infty,0]} + \chi_{[0,\infty]} \Big) \phi' \, \mathrm{d}x \\
&= -\int_{-\infty}^{0} \phi' \, \mathrm{d}x + \int_{0}^{\infty} \phi' \, \mathrm{d}x \\
&= -\Big[\phi \Big]_{-\infty}^{0} + \Big[\phi \Big]_{0}^{\infty} = -2\phi(0).
\end{aligned}$$

The delta distribution (also know as the Dirac distribution) is defined as $\delta(\phi) = \phi(0)$. This calculation shows us that $\partial_x^2 A = -2\delta$. This distribution is not associated to an L^1_{loc} function.

(d) G is linear in φ, so that's okay. We should check the continuity. But this is using the same general idea as (a) and (b): Choose any compact set K and test function supported in K. Then there is a ball $B(0, R)$ that contains K. Then

$$|G(\varphi)| \le \int_{C \cap B(0,R)} \sup_{x \in K} |\phi(x)| \, \mathrm{d}\sigma \le 2\pi \sup_{x \in K} |\phi(x)|.$$

The constant 2π follows since this is the maximum length of the circle C inside the ball $B(0, R)$.

There does not exist such a function g. Suppose for contradiction that it did exist, that $G(\varphi) = F_g(\varphi)$. For every point $y \notin C$ consider a small ball $B(y, r)$ that is disjoint from C. We will now apply Lemma 2.9 to this ball, $\Omega = B(y, r)$. For any test function $\varphi \in C_0^\infty(B(y, r))$ we know that it is zero on C because C and the ball are disjoint:

$$G(\varphi) = \int_C 0 \; d\sigma = 0$$

It follows from the lemma that $g = 0$ on $B(y, r)$ or more generally $g(y) = 0$ for $y \notin C$. But C is a null-set in $\mathbb{R}^2$, so we can say that $g \equiv 0$ as an L^1_{loc} function. This is a contradiction because G is not zero.

20. An induced distribution.

Let $F \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^m)$ and $\psi \in C_0^\infty(\mathbb{R}^m)$. Define

$$\begin{aligned} G : C_0^\infty(\mathbb{R}^n) &\to \mathbb{R}, \\ \varphi &\mapsto F(\varphi \times \psi). \end{aligned}$$

Show that G is a Distribution on $C_0^\infty(\mathbb{R}^n)$, i.e. $G \in D'(\mathbb{R}^n)$. *(3 points)*

Solution. For $\psi \in C_0^\infty(\mathbb{R}^m)$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$, note that $\psi\varphi$ is again a smooth function, and its support it contain in the Cartesian product of the supports of ψ and φ. So we can indeed apply F to the product. As to continuity of G, let $L = \operatorname{supp} \psi \subset \mathbb{R}^m$ and choose a compact set $K \subset \mathbb{R}^n$. For any function $\varphi \in C_0^\infty(K)$, the norm estimate for F gives

$$|G(\varphi)| = |F(\varphi\psi)| \leq C_1 \|\varphi\psi\|_{K \times L, \alpha_1} + \cdots + C_M \|\varphi\psi\|_{K \times L, \alpha_M}.$$

We can also decompose the norms like so

$$\begin{aligned} \|\varphi\psi\|_{K \times L, \alpha} &= \sup_{(x,y) \in K \times L} |\partial^\alpha(\varphi(x)\psi(y))| \\ &= \sup_{(x,y) \in K \times L} \left|\partial^{\alpha'}\varphi(x)\partial^{\alpha''}\psi(y))\right| \\ &\leq \sup_{x \in K} |\partial^{\alpha'}\varphi(x)| \sup_{y \in L} |\partial^{\alpha''}\psi(y))| \\ &= \|\varphi\|_{K,\alpha'} \, \|\psi\|_{L,\alpha''}, \end{aligned}$$

where $\alpha = (\alpha', \alpha'') \in \mathbb{N}_0^{n+m} = \mathbb{N}_0^n \times \mathbb{N}_0^m$ is a decomposition of the multiindex. In this situation, the norms of ψ are fixed constants, so this is a linear combination of the norms of ϕ on K. Substitution of these estimates into the prior bound of $|G(\varphi)|$ gives a bound of the required form.

21. The Crucial Kernel.

When a the partial derivative of a function is zero, it is constant in that direction. In this question we investigate what it means when a distribution has a derivative that is zero. Let $F \in D'(\mathbb{R}^n \times \mathbb{R})$ and let (x, t) with $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ denote the elements in $\mathbb{R}^n \times \mathbb{R}$.
We want to show that: $\partial_t F = 0$ if and only if there is a distribution $G \in D'(\mathbb{R}^n)$ such that

$$F(\varphi) = G\Big(\int_{\mathbb{R}} \varphi(-, t)dt\Big).$$

From a certain point of view then, F does not depend on the t coordinate. In order to show the statement prove the following steps. First, define

$$\begin{aligned} \mathcal{I} : \mathcal{D}(\mathbb{R}^n \times \mathbb{R}) &\to \mathcal{D}(\mathbb{R}^n), \\ \varphi &\mapsto \left(x \mapsto \int_{-\infty}^{\infty} \varphi(x,t)\ \mathrm{d}t \right). \end{aligned}$$

(a) (Optional) Show, that $\mathcal{I}$ is continuous and linear.

(b) Show that a function $\varphi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$ belongs to the kernel of $\mathcal{I}$ if and only if it is the t-derivative of another such function. *(3 points)*

(c) Show that for $F \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$, $\partial_t F = 0$ if and only if $F \equiv 0$ on the kernel of $\mathcal{I}$. *(2 points)*

(d) Finally show the statement by showing that $\partial_t F = 0$ if and only if there exists a $G \in \mathcal{D}'(\mathbb{R}^n)$ with $F(\varphi) = G(\mathcal{I}(\varphi))$. *(2 points)*

Solution.

(a) Linearity follows from linearity of the integral, but perhaps it is good to establish notations:

$$\mathcal{I}(a\varphi + b\psi)(x) = a \int_{-\infty}^{\infty} \varphi(x,t)\ \mathrm{d}t + b \int_{-\infty}^{\infty} \psi(x,t)\ \mathrm{d}t = a\mathcal{I}(\varphi)(x) + b\mathcal{I}(\psi)(x).$$

The function $\mathcal{I}(\varphi)$ is also smooth, because we may pass derivatives through the integral sign.

The question of continuity depends on which norms are being used, and is more subtle. Recall that a linear function is continuous if and only if it is a bounded operator. This explains Definition 2.6. It is enough therefore to bound $\mathcal{I}(\varphi)$ with respect to all of the semi-norms on $\mathcal{D}(\mathbb{R}^n)$. Fix any compact sets $K \subset \mathbb{R}^n$ and $L \subseteq \mathbb{R}^n \times \mathbb{R}$, and choose $\varphi \in C_0^\infty(L)$.

$$\|\mathcal{I}(\varphi)\|_{K,\alpha} = \sup_{x \in K} \left| \partial^\alpha \int_{-\infty}^{\infty} \varphi(x,t)\ \mathrm{d}t \right| \leq \sup_{x \in K} \int_{-\infty}^{\infty} |\partial^\alpha \varphi(x,t)|\ \mathrm{d}t.$$

Now, we don't need to integrate from $-\infty$ to ∞ because φ has compact support. By projecting L to $\mathbb{R}$, we see that there is a bound $T \in \mathbb{R}$ such that if $|t| > T$ then $\varphi(x,t) = 0$ for all $x \in \mathbb{R}^n$.

$$\sup_{x \in K} \int_{-\infty}^{\infty} |\partial^\alpha \varphi(x,t)|\ \mathrm{d}t = \sup_{x \in K} \int_{-T}^{T} |\partial^\alpha \varphi(x,t)|\ \mathrm{d}t \leq 2T \sup_{x \in K} \sup_{t \in [-T,T]} |\partial^\alpha \varphi(x,t)| \leq 2T \|\varphi\|_{L,\alpha}.$$

This shows that $\mathcal{I}$ is a bounded linear operator and therefore is continuous.

(b) Firstly, what does it mean for φ to be in the kernel of $\mathcal{I}$? It means for all $x \in \mathbb{R}^n$

$$\int_{-\infty}^{\infty} \varphi(x,t)\ \mathrm{d}t = 0.$$

Suppose then that φ is in the kernel of $\mathcal{I}$. We must show that there exists $\psi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$ such that $\varphi = \partial_t \psi$. Define

$$\psi(x,t) = \int_{-\infty}^{t} \varphi(x,t)\ \mathrm{d}t.$$

This is a smooth function and its derivative is φ, so it remains to show that it has compact support. As we saw in the previous part, there exists a bound T such that for all $|t| > T$ the function $\varphi(x,t) = 0$ for any $x \in \mathbb{R}^n$. Thus $\psi(x,t) = 0$ for $t < -T$ and $\psi(x,t)$ is a constant for $t > T$. However, the assumption that φ is in the kernel of $\mathcal{I}$ tells us that this constant is zero. Thus ψ also has compact support.

Conversely, take any $\psi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$. Note that

$$\int_{-\infty}^{\infty} \partial_t \psi \, \mathrm{d}t = \psi \Big|_{t=-\infty}^{t=\infty} = 0,$$

so that $\partial_t \psi$ is in the kernel of $\mathcal{I}$.

(c) F is a distribution and $\partial_t F$ means the distributional derivative, ie $\partial_t F(\varphi) = -F(\partial_t \varphi)$. Suppose that $\partial_t F = 0$ and that φ is in the kernel of $\mathcal{I}$. From part (b), we know that $\varphi = \partial_t \psi$ for some $\psi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$. Therefore we apply $\partial_t F$ to ψ to conclude

$$0 = \partial_t F(\psi) = -F(\varphi).$$

This shows that F vanishes on the kernel of $\mathcal{I}$.

In the other direction, suppose that F vanishes on the kernel of $\mathcal{I}$ and take any $\psi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$. Again by part (b), we know that $\partial_t \psi$ is in the kernel of $\mathcal{I}$. Therefore

$$\partial_t F(\psi) = -F(\partial_t \psi) = 0.$$

(d) Before we address F, note that $\mathcal{I}$ is surjective. Explicitly, if $\omega : \mathbb{R} \to \mathbb{R}$ is a function with compact support and $\int_{\mathbb{R}} \omega(t) \, \mathrm{d}t = 1$, and g is any function in $\mathcal{D}(\mathbb{R}^n)$ then $\mathcal{I}(g(x)\omega(t)) = g$. Therefore, as topological vector spaces, $\mathcal{D}(\mathbb{R}^n \times \mathbb{R}) / \ker \mathcal{I}$ and $\mathcal{D}(\mathbb{R}^n)$ are isomorphic.

If $\partial_t F = 0$, from the part (c) we know that F vanishes on $\ker \mathcal{I}$ and so this isomorphism induces a well defined map $G \in \mathcal{D}'(\mathbb{R}^n)$ such that $F(\varphi) = G(\mathcal{I}(\varphi))$. The reverse is immediate: if $F(\varphi) = G(\mathcal{I}(\varphi))$ then F vanishes on the kernel of $\mathcal{I}$ and so must have $\partial_t F = 0$.

22. You can now write "Transport-Distribution Expert" on your résumé.

In this exercise we show that there is a one-to-one correspondence between distributions solving the linear transport equation and distributions describing the corresponding initial values g.

(a) Show that for any distribution $F \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$ which solves the transport equation $(\partial_t + b\nabla)F = 0$, the following distribution solves the equation $\partial_t \tilde{F} = 0$:

$$\tilde{F} \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}) \text{ with } \tilde{F}(\phi) = F(\tilde{\phi}) \text{ and } \tilde{\phi}(y,t) = \phi(y - bt, t) \text{ for all } (y,t) \in \mathbb{R}^n \times \mathbb{R}.$$

(2 points)

(b) Show that for any mollifier $(\lambda_\epsilon)_{\epsilon>0}$ on $\mathbb{R}$ and any $\phi \in C_0^\infty(\mathbb{R}^n)$ the functions

$$\phi \times \lambda_\epsilon : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \quad \text{with} \quad (x,t) \mapsto \phi(x)\lambda_\epsilon(t)$$

belong to $C_0^\infty(\mathbb{R}^n \times \mathbb{R})$. *(1 point)*

(c) Recall $\mathcal{I}$ from the *The Crucial Kernel.* Let $\tilde{F} \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$ solve the equation $\partial_t \tilde{F} = 0$. We have already proved that there exists a distribution $G \in \mathcal{D}(\mathbb{R}^n)$, such that $\tilde{F}(\phi) = G(\mathcal{I}(\phi))$. Argue therefore that $\tilde{F}(\phi \times \lambda_\epsilon)$ does not depend on $\epsilon > 0$. *(1 point)*

(d) Show that for any $G \in \mathcal{D}(\mathbb{R}^n)$ the following $F \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$ solves $(\partial_t + b\nabla)F = 0$:

$$F : C_0^\infty(\mathbb{R}^n \times \mathbb{R}) \to \mathbb{R}, \qquad \qquad \phi \mapsto G\left(\int_{\mathbb{R}} \mathrm{T}_{-tb}\phi(\cdot, t)\ \mathrm{d}t\right),$$

where T_{-tb} is a translation operator. *(3 points)*

(e) Show that $G \to F$ is bijective onto $\{F \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}) \mid (\partial_t + b\nabla)F = 0\}$. *(2 bonus points)*

Solution.

(a) The core of this question is how does the chain rule of differentiation look for distributions? The order of operations is a a little subtle, so to be clear let us write the translation operator $T(\phi) = \phi(y - bt, t)$ explicitly. In other words, $\tilde{F}(\phi) = F(T\phi)$. First observe that T commutes with the spatial derivatives:

$$\partial_k \tilde{\phi} = \partial_k(T\phi) = \partial_k(\phi(x - bt, t)) = T(\partial_k \phi).$$

On the other hand, T does *not* commute with the time derivative. By the chain rule,

$$\partial_t \tilde{\phi} = \partial_t(T\phi) = T(\tilde{\nabla}\phi) \cdot \partial_t T = \begin{pmatrix} T(\nabla\phi) \\ T(\partial_t \phi) \end{pmatrix} \cdot \begin{pmatrix} -b \\ 1 \end{pmatrix} = -b \cdot T(\nabla\phi) + T(\partial_t \phi),$$

where $\tilde{\nabla}$ is the gradient with respect to $\mathbb{R}^n \times \mathbb{R}$ and ∇ is the gradient with respect to $\mathbb{R}^n$. Together this says that

$$T(\partial_t \phi) = \partial_t(T\phi) + b \cdot T(\nabla\phi) = \partial_t(T\phi) + b \cdot \nabla(T\phi).$$

Now we are in a position where we can address the question. In the following we use the definition of the derivative of a distribution and the definition of $\tilde{F}$ and pay close attention to the order of operators:

$$\begin{aligned} \partial_t \tilde{F}(\phi) &= -\tilde{F}(\partial_t \phi) = -F(T(\partial_t \phi)) = -F(\partial_t(T\phi) + b \cdot \nabla(T\phi)) \\ &= -F(\partial_t(T\phi)) - F(b \cdot \nabla(T\phi)) = \partial_t F(T\phi) + b \cdot \nabla F(T\phi) \\ &= (\partial_t + b \cdot \nabla)F(T\phi) = 0. \end{aligned}$$

(b) The product of two smooth functions is smooth. So it only remains to show that the product has compact support. Let K is the support of ϕ and the support of λ_ϵ is I. If $(x, t) \notin K \times I$ then either $\phi(x) = 0$ or $\lambda_\epsilon(t) = 0$ (or both). In both cases the product is zero. This shows that the support of the product is contained in $K \times I$, which is a bounded set, and thus the support of the product must be compact.

(c) We compute

$$\mathcal{I}(\phi \times \lambda_\epsilon)(x) = \int_{-\infty}^{\infty} \phi(x)\lambda_\epsilon(t)\ \mathrm{d}t = \phi(x) \int_{-\infty}^{\infty} \lambda_\epsilon(t)\ \mathrm{d}t = \phi(x),$$

because the integral of a mollifier is always 1. In other words, $\mathcal{I}(\phi \times \lambda_\epsilon) = \phi$. As explained the question, the condition that $\partial_t \tilde{F} = 0$ means that it is of the form $\tilde{F}(\psi) = G(\mathcal{I}(\psi))$ for some distribution G. Therefore $\tilde{F}(\phi \times \lambda_\epsilon) = G(\mathcal{I}(\phi \times \lambda_\epsilon)) = G(\phi)$ is independent of ϵ.

(d) Again, the order of operators in this question is somewhat subtle. Let us introduce a translation $S(\phi) = \phi(x + bt, t)$. This is similar to T from part (a), in fact they are inverses, and we have that S commutes with ∇ but

$$S(\partial_t \phi) = \partial_t(S\phi) - b \cdot S(\nabla\phi).$$

One could also write the integral part of this formula using the operator $\mathcal{I}$, but we don't have to interchange its position, so we will leave it as an integral so as not to be more abstract than necessary. Perhaps it would be a good exercise to rewrite the following proof using $\mathcal{I}$.

In this notation we have that

$$F(\phi) := G\left(x \mapsto \int_{\mathbb{R}} S\phi \, \mathrm{d}t\right).$$

Let us compute the t-derivative of this F: for any test function ϕ,

$$\begin{aligned}
\partial_t F(\phi) = -F(\partial_t \phi) &= -G\left(x \mapsto \int_{\mathbb{R}} S(\partial_t \phi) \, \mathrm{d}t\right) \\
&= -G\left(x \mapsto \int_{\mathbb{R}} \partial_t(S\phi) - b \cdot S(\nabla\phi) \, \mathrm{d}t\right) \\
&= -G\left(x \mapsto 0 - \int_{\mathbb{R}} b \cdot S(\nabla\phi) \, \mathrm{d}t\right) \\
&= \sum_{k=1}^{n} b_k G\left(x \mapsto \int_{\mathbb{R}} S(\partial_k \phi) \, \mathrm{d}t\right) \\
&= \sum_{k=1}^{n} b_k F(\partial_k \phi) = -b \cdot \nabla F(\phi).
\end{aligned}$$

This shows that it solves the transport equation.

(e) Part (d) shows that the mapping $G \mapsto F$ is well-defined. Suppose then that we had a solution F of the transport equation. Part (a) shows there is an associated distribution $\tilde{F}$ with the property that $\partial_t \tilde{F} = 0$. Using part (c) we have $\tilde{F}(\phi) = G(\mathcal{I}(\phi))$ for some $G \in \mathcal{D}'(\mathbb{R}^n)$. This gives a mapping $F \mapsto G$.

It remains to show that these mappings are inverse to one another, but observe

$$F(\phi) = F(TS\phi) = \tilde{F}(S\phi) = G(\mathcal{I}(S\phi)),$$

which crucially relies on T and S being inverse translations. The mapping is therefore bijective.