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Introduction to Partial Differential Equations
Exercise sheet 6

11th October, 2022

19. Distributions.

(a) Choose any compact set $K \subset \mathbb{R}$. Since it is bounded, there exists $R > 0$ with $K \subseteq [-R, R]$. Now choose any test function $\phi \in C_0^\infty(\mathbb{R})$ with compact support in K. Since it is continuous, $\sup_{x \in K} |\phi(x)|$ is finite. Prove the following inequality

$$\left| \int_{-\infty}^{\infty} |x|\phi(x) \, \mathrm{d}x \right| \leq R^2 \sup_{x \in K} |\phi(x)|$$

(2 points)

(b) Show directly from Definition 2.6 that the distribution associated to the absolute value function

$$A : C_0^\infty(\mathbb{R}) \to \mathbb{R}, \; \phi \mapsto \int_{-\infty}^{\infty} |x|\phi(x) \, \mathrm{d}x$$

is in fact a distribution on $\mathbb{R}$. *(1 point)*

(c) Calculate and describe the first and second derivatives of A as a distribution.

(3 points)

(d) Consider the circle $C = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$. Show that

$$G(\varphi) := \int_C \varphi \, d\sigma$$

defines a distribution in $\mathcal{D}'(\mathbb{R}^2)$. Note that the $d\sigma$ indicates this is an integration over the submanifold C. Does there exist a locally integrable function $g : \mathbb{R}^2 \to \mathbb{R}$ with

$$G(\varphi) = \int_{\mathbb{R}^2} g \, \varphi \, \mathrm{d}x$$

for all $\varphi \in C_0^\infty(\mathbb{R})$? (Hint. Use Lemma 2.9) *(2 Points + 2 Bonus Points)*

20. An induced distribution.

Let $F \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^m)$ and $\psi \in C_0^\infty(\mathbb{R}^m)$. Define

$$G : C_0^\infty(\mathbb{R}^n) \to \mathbb{R},$$
$$\varphi \mapsto F(\varphi \times \psi).$$

Show that G is a Distribution on $C_0^\infty(\mathbb{R}^n)$, i.e. $G \in D'(\mathbb{R}^n)$. *(3 points)*

21. The Crucial Kernel.

When a the partial derivative of a function is zero, it is constant in that direction. In this question we investigate what it means when a distribution has a derivative that is zero. Let $F \in D'(\mathbb{R}^n \times \mathbb{R})$ and let (x, t) with $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ denote the elements in $\mathbb{R}^n \times \mathbb{R}$.
We want to show that: $\partial_t F = 0$ if and only if there is a distribution $G \in D'(\mathbb{R}^n)$ such that

$$F(\varphi) = G\Big(\int_{\mathbb{R}} \varphi(-, t) dt \Big).$$

From a certain point of view then, F does not depend on the t coordinate. In order to show the statement prove the following steps. First, define

$$\begin{aligned} \mathcal{I} : \mathcal{D}(\mathbb{R}^n \times \mathbb{R}) &\to \mathcal{D}(\mathbb{R}^n), \\ \varphi &\mapsto \left(x \mapsto \int_{-\infty}^{\infty} \varphi(x,t) \, \mathrm{d}t \right). \end{aligned}$$

(a) (Optional) Show, that $\mathcal{I}$ is continuous and linear.

(b) Show that a function $\varphi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$ belongs to the kernel of $\mathcal{I}$ if and only if it is the t-derivative of another such function. *(3 points)*

(c) Show that for $F \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$, $\partial_t F = 0$ if and only if $F \equiv 0$ on the kernel of $\mathcal{I}$. *(2 points)*

(d) Finally show the statement by showing that $\partial_t F = 0$ if and only if there exists a $G \in \mathcal{D}'(\mathbb{R}^n)$ with $F(\varphi) = G(\mathcal{I}(\varphi))$. *(2 points)*

22. You can now write "Transport-Distribution Expert" on your résumé.

In this exercise we show that there is a one-to-one correspondence between distributions solving the linear transport equation and distributions describing the corresponding initial values g.

(a) Show that for any distribution $F \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$ which solves the transport equation $(\partial_t + b\nabla)F = 0$, the following distribution solves the equation $\partial_t \tilde{F} = 0$:

$$\tilde{F} \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}) \text{ with } \tilde{F}(\phi) = F(\tilde{\phi}) \text{ and } \tilde{\phi}(y,t) = \phi(y - bt, t) \text{ for all } (y,t) \in \mathbb{R}^n \times \mathbb{R}.$$

(2 points)

(b) Show that for any mollifier $(\lambda_\epsilon)_{\epsilon>0}$ on $\mathbb{R}$ and any $\phi \in C_0^\infty(\mathbb{R}^n)$ the functions

$$\phi \times \lambda_\epsilon : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \quad \text{with} \quad (x,t) \mapsto \phi(x)\lambda_\epsilon(t)$$

belong to $C_0^\infty(\mathbb{R}^n \times \mathbb{R})$. *(1 point)*

(c) Recall $\mathcal{I}$ from the *The Crucial Kernel*. Let $\tilde{F} \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$ solve the equation $\partial_t \tilde{F} = 0$. We have already proved that there exists a distribution $G \in \mathcal{D}(\mathbb{R}^n)$, such that $\tilde{F}(\phi) = G(\mathcal{I}(\phi))$. Argue therefore that $\tilde{F}(\phi \times \lambda_\epsilon)$ does not depend on $\epsilon > 0$. *(1 point)*

(d) Show that for any $G \in \mathcal{D}(\mathbb{R}^n)$ the following $F \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$ solves $(\partial_t + b\nabla)F = 0$:

$$F : C_0^\infty(\mathbb{R}^n \times \mathbb{R}) \to \mathbb{R}, \qquad \phi \mapsto G\left(\int_{\mathbb{R}} \mathrm{T}_{-tb}\phi(\cdot,t) \, \mathrm{d}t \right),$$

where T_{-tb} is a translation operator. *(3 points)*

(e) Show that $G \to F$ is bijective onto $\{F \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}) \mid (\partial_t + b\nabla)F = 0\}$. *(2 bonus points)*