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**Introduction to Partial Differential Equations**
**Exercise sheet 5**

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**13. The Music of the Spheres.**

In this exercise we show the connection between integration over a ball and integration on spheres in $\mathbb{R}^{n+1}$. More precisely

$$\int_{B(0,R)} f(x)\, dx = \int_0^R \left( \int_{\partial B(0,r)} f(x)\, d\sigma(x) \right) dr$$

You may answer this question in full generality for $\mathbb{R}^{n+1}$ or just for $\mathbb{R}^2$, your choice.

**(a)** Suppose that you have an $(n+1) \times n$ matrix $A$ and a unit vector $b \in \mathbb{R}^{n+1}$ such that $b$ is perpendicular to every column of $A$. That is $b^T A = 0$. Let $\tilde{A} = (b \mid A)$ be the square matrix with $b$ as the first column. Argue that *(2 bonus points)*

$$(\det \tilde{A})^2 = \det \tilde{A}^T \tilde{A} = \det A^T A.$$

**(b)** Let $\Phi : U \to \partial B(0,1)$ be a parameterisation of the unit sphere. Observe then that $\Psi : [0,R] \times U \to B(0,R)$ with $\Psi(r,\theta) = r\Phi(\theta)$ is a parameterisation of the ball. Show that the change of variables matrix for $\Psi$ in the integral on the left hand side above has the form of $\tilde{A}$. *(2 points)*

**(c)** Hence prove the integration formula. *(2 points)*

**Solution.**

**(a)** The first equality holds because $\tilde{A}$ and $\tilde{A}^T$ have the same determinant and determinant is multiplicative. Consider then

$$\tilde{A}^T \tilde{A} = \begin{pmatrix} b^T \\ A^T \end{pmatrix} \begin{pmatrix} b & A \end{pmatrix} = \begin{pmatrix} b^T b & b^T A \\ (b^T A)^T & A^T A \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & A^T A \end{pmatrix}.$$

This is in block diagonal form, so the result follows.

**(b)** The change of variables matrix is

$$\Psi' = \begin{pmatrix} \frac{\partial \Psi}{\partial r} & \frac{\partial \Psi}{\partial \theta_1} & \cdots \end{pmatrix}.$$

By the definition of $\Psi = r\Phi$, it follows $\frac{\partial \Psi}{\partial r} = \Phi$, a unit vector. On the other hand, the relation $\Phi \cdot \Phi = 1$ shows that $\frac{\partial \Psi}{\partial \theta_i} = r\frac{\partial \Phi}{\partial \theta_i}$ is perpendicular to $\Phi$. Thus $\Psi'$ does have the required form.

**(c)** If $\Phi$ is a parameterisation of the unit sphere, then $r\Phi$ is a parameterisation of $\partial B(0,r)$. In the definition of surface integral, the area element is

$$\sqrt{\det(r\Phi')^T(r\Phi')} = \sqrt{\det(\Psi')^T \Psi'} = |\det \Psi'|$$

using the previous part. With this prepared, we compute

$$\begin{aligned}
\int_{B(0,R)} f(x)\,dx &= \int_{[0,R]\times U} f\circ\Psi\,|\det\Psi'|\,dy \\
&= \int_0^R \int_U f\circ\Psi\,|\det\Psi'|\,d\theta\,dr \\
&= \int_0^R \int_U f\circ(r\Phi)\,\sqrt{\det(r\Phi')^T(r\Phi')}\,d\theta\,dr \\
&= \int_0^R \left(\int_{\partial B(0,r)} f(x)\,d\sigma(x)\right) dr.
\end{aligned}$$

**14. In Colour.**

Let $\Omega$ be a region in $\mathbb{R}^n$ and $N$ the outer unit normal vector field on $\partial\Omega$. Let $u, v$ be two $C^2$ real-valued functions on $\overline{\Omega}$.

**(a)** Prove the first Green formula

$$\int_\Omega v\triangle u\,dx = -\int_\Omega \nabla u\cdot\nabla v\,dx + \int_{\partial\Omega} v\nabla u\cdot N\,d\sigma.$$

*(3 points)*

**(b)** Using the first Green formula, prove the second Green formula

$$\int_\Omega (v\triangle u - u\triangle v)\,dx = \int_{\partial\Omega} (v\nabla u - u\nabla v)\cdot N\,d\sigma.$$

*(1 point)*

**(c)** Suppose further that $v$ has compact support in $\Omega$. Prove that

$$\int_\Omega v\triangle u\,dx = \int_\Omega u\triangle v\,dx$$

*(1 point)*

**Solution.**

**(a)** The final term of the formula can have the divergence theorem applied to it:

$$-\int_\Omega \nabla u\cdot\nabla v\,dx + \int_{\partial\Omega} v\nabla u\cdot N\,d\sigma = \int_\Omega -\nabla u\cdot\nabla v + \nabla\cdot(v\nabla u)\,dx$$

If $g$ is a scalar-valued function and $F$ is a vector-valued function, recall the product rule (or derive it for yourself): $\nabla\cdot(gF) = (\nabla g)\cdot F + g\nabla\cdot F$. Applying this formula here, and remembering that the Laplacian is the divergence of the gradient, gives

$$\int_\Omega -\nabla u\cdot\nabla v + \nabla\cdot(v\nabla u)\,dx = \int_\Omega -\nabla u\cdot\nabla v + \nabla v\cdot\nabla u + v\Delta u\,dx = \int_\Omega v\Delta u\,dx.$$

**(b)** The second Greens formula is simply a symmetrised version of the first:

$$\begin{aligned}\int_{\Omega}(v\triangle u - u\triangle v)\,dx &= -\int_{\Omega}(\nabla u\cdot\nabla v - \nabla v\cdot\nabla u)\,dx + \int_{\partial\Omega}(v\nabla u - u\nabla v)\cdot N\,d\sigma\\ &= \int_{\partial\Omega}(v\nabla u - u\nabla v)\cdot N\,d\sigma.\end{aligned}$$

**(c)** Since $v$ has compact support, it and its derivatives must vanish on $\partial\Omega$. Therefore the right hand side of (b) is zero. The result follows.

**15. The Black Spot.**

Consider the plane $\mathbb{R}^2$, a disc $B_r = \{x^2+y^2 \le r^2\}$ and the function $g(x,y) = \ln(x^2+y^2)$.

**(a)** Show that the value of the integral

$$\int_{\partial B_r} \nabla g\cdot N\,d\sigma$$

does not depend on the radius $r$, where $N$ is the outward pointing normal. *(2 points)*

**(b)** What property of $g$ explains this fact? In your proof, be careful to note that $g$ is singular at $(0,0)$. *(3 points)*

**(c)** Prove for any compact region $\Omega\subset\mathbb{R}^2$ whose boundary is a manifold, that

$$\int_{\partial\Omega}\nabla g\cdot N\,d\sigma = \begin{cases}4\pi & \text{if } (0,0) \text{ lies in the interior of } \Omega\\ 0 & \text{if } (0,0) \text{ lies in the exterior of } \Omega\end{cases}$$

*(2 points)*

**Solution.**

**(a)** The outward pointing unit normal of the ball of radius $r$ is $N = r^{-1}(x,y)$, where $r = \sqrt{x^2+y^2}$ is constant on $\partial B_r$. The gradient $\nabla g = r^{-2}(2x,2y)$. Together we then have

$$\int_{\partial B_r}\nabla g\cdot N\,d\sigma = \int_{\partial B_r} r^{-3}(2x^2+2y^2)\,d\sigma = 2r^{-1}\int_{\partial B_r} 1\,d\sigma.$$

Now, the integral above is just the 1 times the circumference of the circle, and therefore the value is $2r^{-1}\times 2\pi r = 4\pi$.

**(b)** Let us consider the difference of two of these integrals for different radii $r<R$.

$$\int_{\partial B_R}\nabla g\cdot N\,d\sigma - \int_{\partial B_r}\nabla g\cdot N\,d\sigma = \int_{\partial B_R}\nabla g\cdot N\,d\sigma + \int_{\partial B_r}\nabla g\cdot(-N)\,d\sigma = \int_{\partial A_{r,R}}\nabla g\cdot N\,d\sigma,$$

where $A_{r,R}$ is the annulus with inner radius $r$ and outer radius $R$. Note, the boundary of the annulus consists of two disjoint circles and the outward pointing normal of the annulus on the inner boundary circle is the outward point normal of the disc $B_r$. This explains the sign in the above calculation.

If we apply the divergence theorem to the annulus, we get

$$\int_{\partial A_{r,R}} \nabla g \cdot N \, d\sigma = \int_{A_{r,R}} \Delta g \, dx,$$

and the Laplacian of $g$ is zero:

$$\begin{aligned}
\Delta g &= \frac{\partial}{\partial x} \frac{2x}{x^2+y^2} + \frac{\partial}{\partial y} \frac{2y}{x^2+y^2} \\
&= \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2} + \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2} \\
&= 0.
\end{aligned}$$

Therefore the integral over the annulus is zero, and hence the difference of the integrals on the two circles is also zero.

If we are being precise, the Laplacian of $g$ is zero on $\mathbb{R}^2 \setminus \{(0,0)\}$; at the origin it is not defined/singular. This is why we could not apply the divergence theorem directly to $B_r$.

**(c)** Suppose first that $(0,0) \notin \Omega$. We have already seen that the Laplacian of $g$ is zero, and we can immediately conclude that the integral over $\partial\Omega$ is zero too.

If however $(0,0)$ lies in the interior of $\Omega$, we must first excise it. There exists some small $\epsilon$ such that $B_\epsilon \subset \Omega$. Then

$$\begin{aligned}
\int_{\partial\Omega} \nabla g \cdot N \, d\sigma &= \int_{\partial B_\epsilon} \nabla g \cdot N \, d\sigma + \left( \int_{\partial\Omega} \nabla g \cdot N \, d\sigma - \int_{\partial B_\epsilon} \nabla g \cdot N \, d\sigma \right) \\
&= 4\pi + \int_{\Omega \setminus B_\epsilon} \Delta g \, dx \\
&= 4\pi.
\end{aligned}$$

**16. Convoluted.**

The convolution of two functions $f, g : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(y) g(x-y) \, dy.$$

**(a)** Let $f_n(x) = 0.5n$ for $x \in [-n^{-1}, n^{-1}]$ and 0 otherwise. Show that the following bounds hold

$$\inf_{|y| \leq n^{-1}} g(y) \leq (g * f_n)(0) \leq \sup_{|y| \leq n^{-1}} g(y).$$

*(3 points)*

**(b)** Suppose now that $g$ is continuous. Show that $(g * f_n)(0) \to g(0)$ as $n \to \infty$. *(3 points)*

**(c)** (Optional) Show that the convolution of $C_0^\infty$-functions on $\mathbb{R}^n$ is a bilinear, commutative, and associative operation.

**Solution.**

**(a)** We compute

$$\begin{aligned}(g * f_n)(0) &= \int_{-\infty}^{\infty} g(y) f_n(-y) = \int_{-n^{-1}}^{n^{-1}} g(y) \times \frac{1}{2} n \\ &\leq \frac{1}{2} n \int_{-n^{-1}}^{n^{-1}} \sup_{0 \leq y \leq n^{-1}} g(y) = \frac{1}{2} n \sup_{|y| \leq n^{-1}} g(y) \times 2n^{-1} \\ &= \sup_{|y| \leq n^{-1}} g(y)\end{aligned}$$

In a similar manner, we see that $(g * f_n)(0)$ is bound below by $\inf_{|y| \leq n^{-1}} g(y)$.

**(b)** Clearly $\sup_{|y|<n^{-1}} g(y) \geq g(0)$. On the other hand, choose any $\epsilon > 0$. By the continuity of $g$, there exists $\delta > 0$ such that $|g(y) - g(0)| < \epsilon$ for all $y \in (-\delta, \delta)$. Choose $N$ such that $N^{-1} < \delta$. That means for all $|y| < N^{-1}$ we have $|g(y) - g(0)| < \epsilon$. For all $n > N$ the interval $[-n^{-1}, n^{-1}]$ is a subset of $[-N^{-1}, N^{-1}]$. It follows that

$$\sup_{|y|<n^{-1}} g(y) \leq \sup_{|y|<N^{-1}} g(y) < \sup_{|y|<N^{-1}} (g(0) + \epsilon) = g(0) + \epsilon.$$

These two inequalities together say that $\forall \epsilon > 0 \, \exists N \, \forall n > N$ it holds that

$$|\sup_{y \in I_n} g(y) - g(0)| < \epsilon.$$

This is the definition of $\sup_{|y|<n^{-1}} g(y) \to g(0)$. The same argument shows that $\inf_{|y|<n^{-1}} g(y) \to g(0)$ also. By the sandwich rule/squeeze rule, the result follows.

**(c)** Formal bilinearity follows from the linearity of the integral and the bilinearity of the product of functions. The smoothness and compact support of all functions involved means that the integrals always exist.

Commutativity:

$$f * g(x) = \int_{-\infty}^{\infty} f(y) g(x-y) \, dy = \int_{\infty}^{-\infty} f(x-z) g(z) \, (-dz) = g * f(x).$$

Associativity

$$\begin{aligned}f * (g * h)(x) &= \int_{-\infty}^{\infty} f(y)(g * h)(x-y) \, dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) g(z) h((x-y)-z) \, dz \, dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) g(z) h(x-(y+z)) \, dz \, dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) g(w-y) h(x-w) \, dy \, dw \\ &= \int_{-\infty}^{\infty} (f * g)(w) h(x-w) \, dw \\ &= (f * g) * h(x)\end{aligned}$$

**17. Is this an applied math course?**

In economics, the Black-Scholes equation is a PDE that describes the price $V$ of a (European-style) option which under some assumptions about the risk and expected return, as a function of time $t$ and current stock price $S$. The equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S},$$

where $r$ and $\sigma$ are constants representing the interest rate and the stock volatility respectively. Describe the order of this equation, and whether it is elliptic, parabolic, and/or hyperbolic.

*(3 points)*

**Solution.** The highest derivative in the equation is second order, so this is a second order PDE. To determine which type it is, bring all the terms to one side and order them as per the general form of a second order linear PDE

$$\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + 0\frac{\partial^2 V}{\partial S \partial t} + 0\frac{\partial^2 V}{\partial t \partial S} + 0\frac{\partial^2 V}{\partial t^2}\right) + \left(rS\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t}\right) - rV = 0.$$

From this we can read off all the coefficient functions:

$$a = \begin{pmatrix} \frac{1}{2}\sigma^2 S^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} rS \\ 1 \end{pmatrix}, \quad c = -r.$$

We see that the matrix $a$ is positive semi-definite and in fact its kernel is one dimensional. That makes it a parabolic PDE.

**18. Go with the flow.**

(Optional extra question)

In this question we generalise the conservation law to the form usually encountered in physics. Let $\rho(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be the density of a substance. We have seen in an earlier question that the flux density is simply the density multiplied by the velocity $\rho v$, for a velocity field $v(x,t) : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$. The flux across a $(n-1)$-dimensional submanifold $S$ is the integral

$$\int_S \rho v \cdot N \, d\sigma,$$

where $N$ is the normal of $S$.

**(a)** Argue that the conservation of substance is equivalent to

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0.$$

This is the usual form of the conservation law in physics.

**(b)** How does this relate to the form of the conservation law derived in the lectures?

**(c)** For liquids a common property is *incompressibility*. For example, water is well-modelled as an incompressible liquid (at the bottom of the ocean, it is compressed by just 2%). Normally this would imply that $\rho$ is constant. However, slightly more general model says that $\rho$ is not globally constant, but if we follow a point $x(t)$ along the velocity field $v$ then $\rho(x(t), t)$ is constant.
Use this description of incompressible flow to show that $\nabla \cdot v = 0$.

**Solution.**

**(a)** By defining flux in the way we have, the divergence theorem applies. Let $S$ be a surface enclosing a volume $V$:

$$\int_S \rho v \cdot N \, d\sigma = \int_V \nabla \cdot (\rho v) \, dx.$$

On the other hand, the amount of substance in $V$ is the integral of $\rho$. Conservation means that the (positive) change of substance should be equal to the negative of the outward flux:

$$\frac{\partial}{\partial t} \int_V \rho \, dx = -\int_S \rho v \cdot N \, d\sigma \Rightarrow \int_V \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) \, dx = 0.$$

Since this should hold for every volume $V$, we conclude that the integrand must be identically zero. We will give a rigorous proof of this final step in a few weeks time.

**(b)** The conservation law in lectures was only for one-dimensional situations, so $\nabla \cdot$ is the same as $\partial_x$. In the previous part we could also have an arbitrary flux function $f$ instead of only the form $\rho v$ and still apply the same working to arrive at

$$\frac{\partial \rho}{\partial t} + \nabla \cdot f = 0 \Rightarrow \dot{\rho} + \partial_x f = 0.$$

If the flux function only depends on the density $\rho$, and not on the coordinate, then we arrive exactly at the form in Theorem 1.10.

**(c)** The condition of being constant along a flow is very similar to the idea behind the method of characteristics. If we take the total derivative

$$0 = \frac{d}{dt}\rho = \nabla \rho \cdot \dot{x} + \dot{\rho} = \nabla \rho \cdot v + \dot{\rho}.$$

On the other hand, we can expand the conservation law

$$0 = \dot{\rho} + \nabla \cdot (\rho v) = \dot{\rho} + \nabla \rho \cdot v + \rho \nabla \cdot v = \rho \nabla \cdot v.$$

So either there is no substance (which is trivial) or $\nabla \cdot v = 0$ as required.