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Introduction to Partial Differential Equations
Exercise sheet 5

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13. The Music of the Spheres.

In this exercise we show the connection between integration over a ball and integration on spheres in $\mathbb{R}^{n+1}$. More precisely

$$\int_{B(0,R)} f(x)\; dx = \int_0^R \left(\int_{\partial B(0,r)} f(x)\; d\sigma(x) \right) dr$$

You may answer this question in full generality for $\mathbb{R}^{n+1}$ or just for $\mathbb{R}^2$, your choice.

(a) Suppose that you have an $(n+1) \times n$ matrix A and a unit vector $b \in \mathbb{R}^n$ such that b is perpendicular to every column of A. That is $b^T A = 0$. Let $\tilde{A} = (b \mid A)$ be the square matrix with b as the first column. Argue that *(2 bonus points)*

$$(\det \tilde{A})^2 = \det \tilde{A}^T \tilde{A} = \det A^T A.$$

(b) Let $\Phi : U \to \partial B(0,1)$ be a parameterisation of the unit sphere. Observe then that $\Psi : [0,R] \times U \to B(0,R)$ with $\Psi(r,\theta) = r\Phi(\theta)$ is a parameterisation of the ball. Show that the change of variables matrix for Ψ in the integral on the left hand side above has the form of $\tilde{A}$. *(2 points)*

(c) Hence prove the integration formula. *(2 points)*

14. In Colour.

Let Ω be a region in $\mathbb{R}^n$ and N the outer unit normal vector field on $\partial\Omega$. Let u, v be two C^2 real-valued functions on $\overline{\Omega}$.

(a) Prove the first Green formula

$$\int_\Omega v \triangle u\; dx = -\int_\Omega \nabla u \cdot \nabla v\; dx + \int_{\partial\Omega} v \nabla u \cdot N\; d\sigma.$$

(3 points)

(b) Using the first Green formula, prove the second Green formula

$$\int_\Omega (v\triangle u - u\triangle v)\; dx = \int_{\partial\Omega} (v\nabla u - u\nabla v) \cdot N\; d\sigma.$$

(1 point)

(c) Suppose further that v has compact support in Ω. Prove that

$$\int_\Omega v\triangle u\; dx = \int_\Omega u \triangle v\; dx$$

(1 point)

15. The Black Spot.

Consider the plane $\mathbb{R}^2$, a disc $B_r = \{x^2 + y^2 \leq r^2\}$ and the function $g(x, y) = \ln(x^2 + y^2)$.

(a) Show that the value of the integral

$$\int_{\partial B_r} \nabla g \cdot N \, d\sigma$$

does not depend on the radius r, where N is the outward pointing normal. *(2 points)*

(b) What property of g explains this fact? In your proof, be careful to note that g is singular at $(0, 0)$. *(3 points)*

(c) Prove for any compact region $\Omega \subset \mathbb{R}^2$ whose boundary is a manifold, that

$$\int_{\partial\Omega} \nabla g \cdot N \, d\sigma = \begin{cases} 4\pi & \text{if } (0,0) \text{ lies in the interior of } \Omega \\ 0 & \text{if } (0,0) \text{ lies in the exterior of } \Omega \end{cases}$$

(2 points)

16. Convoluted.

The convolution of two functions $f, g : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(y)g(x - y) \, dy.$$

(a) Let $f_n(x) = 0.5n$ for $x \in [-n^{-1}, n^{-1}]$ and 0 otherwise. Show that the following bounds hold

$$\inf_{|y| \leq n^{-1}} g(y) \leq (g * f_n)(0) \leq \sup_{|y| \leq n^{-1}} g(y).$$

(3 points)

(b) Suppose now that g is continuous. Show that $(g * f_n)(0) \to g(0)$ as $n \to \infty$. *(3 points)*

(c) (Optional) Show that the convolution of C_0^∞-functions on $\mathbb{R}^n$ is a bilinear, commutative, and associative operation.

17. Is this an applied math course?

In economics, the Black-Scholes equation is a PDE that describes the price V of a (European-style) option which under some assumptions about the risk and expected return, as a function of time t and current stock price S. The equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S},$$

where r and σ are constants representing the interest rate and the stock volatility respectively. Describe the order of this equation, and whether it is elliptic, parabolic, and/or hyperbolic.

(3 points)

18. Go with the flow.

(Optional extra question)

In this question we generalise the conservation law to the form usually encountered in physics. Let $\rho(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be the density of a substance. We have seen in an earlier question that the flux density is simply the density multiplied by the velocity ρv, for a velocity field $v(x,t) : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$. The flux across a $(n-1)$-dimensional submanifold S is the integral

$$\int_S \rho v \cdot N \, d\sigma,$$

where N is the normal of S.

(a) Argue that the conservation of substance is equivalent to

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0.$$

This is the usual form of the conservation law in physics.

(b) How does this relate to the form of the conservation law derived in the lectures?

(c) For liquids a common property is *incompressibility*. For example, water is well-modelled as an incompressible liquid (at the bottom of the ocean, it is compressed by just 2%). Normally this would imply that ρ is constant. However, slightly more general model says that ρ is not globally constant, but if we follow a point $x(t)$ along the velocity field v then $\rho(x(t),t)$ is constant.
Use this description of incompressible flow to show that $\nabla \cdot v = 0$.