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Introduction to Partial Differential Equations
Exercise sheet 4

27th September, 2022

10. Solving PDEs Solve the initial value problems of the following PDEs using the method of characteristics. You may assume that g is continuously differentiable on the corresponding domain.

(a) $x_1\partial_2 u - x_2\partial_1 u = u$ on the domain $x_1, x_2 > 0$, with initial condition $u(x_1, 0) = g(x_1)$.
(4 points)

(b) $x_1\partial_1 u + 2x_2\partial_2 u + \partial_3 u = 3u$ on $x_1, x_2 \in \mathbb{R}$, $x_3 > 0$, with initial condition $u(x_1, x_2, 0) = g(x_1, x_2)$.
(4 points)

(c) $u\partial_1 u + \partial_2 u = 1$ on the domain $x_1, x_2 > 0$, with initial condition $u(x_1, x_1) = \frac{1}{2}x_1$.
(5 points)

Solution. These PDEs are all of the linear type of the previous question, so we can use the ODEs for the characteristics that we have already derived in the exercise 'Linear Partial Differential Equations', which is also the notation of the general method of characteristics.

(a) This PDE is $(-x_2, x_1) \cdot p - z = 0$. The system of ODEs therefore reads in part

$$\dot{x}_1 = -x_2, \quad \dot{x}_2 = x_1,$$

which is the well know system solved by the sinusoidal functions. From the boundary condition $(x_1, 0)$, we should choose $x_{20} = 0$. Therefore $x_2 = x_{10}\sin s$ and $x_1 = x_{10}\cos s$ are the characteristics. Given a point (x_1, x_2) we can determine the parameters of the characteristics as

$$s = \arctan(x_2/x_1), \;\; x_{10} = \sqrt{x_1^2 + x_2^2} = |x|.$$

The ODE describing the values of u is $\dot{z} = z$, which has the solution $z(s) = e^s z(0)$. From the initial condition

$$z(0) = u(x(0)) = u(x_{10}, 0) = g(x_{10}).$$

Putting this together

$$u(x) = e^{\arctan(x_2/x_1)} g(|x|).$$

(b) From $F = (x_1, 2x_2, 1) \cdot p - 3z$ it follows that

$$x(s) = (x_{10}e^s, x_{20}e^{2s}, x_{30} + s) = (x_{10}e^s, x_{20}e^{2s}, s),$$

where we choose our starting points for the characteristics to lie in the case of the initial conditions, which requires us to set $x_{30} = 0$. Already we can determine the appropriate parameter values for any point: $s = x_3$, $x_{10} = x_1 e^{-x_3}$, and $x_{20} = x_2 e^{-2x_3}$. The the equation for the values is $z = z(0)e^{3s}$, so

$$u(x) = e^{3s}u(x_{10}, x_{20}, 0) = e^{3x_3} g\left(x_1 e^{-x_3}, x_2 e^{-2x_3}\right).$$

(c) This PDE, $F = (z, 1) \cdot p - 1$ is a little different to the others, because of the z in the coefficients of p. This creates a linkage in the system of ODEs:

$$\dot{x}_1 = z, \quad \dot{x}_2 = 1, \quad \dot{z} = 1.$$

Fortunately, we can solve for z first this time quite easily: $z(s) = s + z(0)$. Then $x(s) = (\frac{1}{2}s^2 + sz(0) + x_{10}, s + x_{20})$. Choose $x_{20} = x_{10}$. This choice means that $x_{10} = x_2 - s$. The initial conditions give $z(0) = u(x(0)) = u(x_{10}, x_{10}) = \frac{1}{2}x_{10}$. Together this allows us to solve for s:

$$\begin{aligned} x_1 &= \frac{1}{2}s^2 + s\frac{1}{2}(x_2 - s) + x_2 - s \\ x_1 - x_2 &= \frac{1}{2}x_2 s - s \\ s &= \frac{2x_1 - 2x_2}{x_2 - 2}. \end{aligned}$$

Finally, what we are interested in is the value of the solution u on these curves, and $u(x(s)) = z(s) = s + z(0) = s + \frac{1}{2}(x_2 - s)$, ie

$$u(x) = \frac{1}{2}x_2 + \frac{1}{2}\frac{2x_1 - 2x_2}{x_2 - 2}.$$

11. Duhamel's Principle

Duhamel's principle occurs in a few places in the script. In this exercise we give the general idea and show how it applies to the transport equation. It is a method to solve an inhomogeneous PDE on $\mathbb{R}^n \times \mathbb{R}$ of the following form

$$\partial_t u - Lu = f(x, t), \quad u(x, 0) = 0,$$

where L is a linear differential operator on $\mathbb{R}^n$ with constant coefficients. The idea is to instead we consider the following family of homogeneous equations

$$\partial_t u_s - Lu_s = 0, \quad u_s(x, s) = f(x, s).$$

Suppose that we can find such solutions u_s. Prove that

$$u(x, t) = \int_0^t u_s(x, t)\, ds$$

is a solution to the inhomogeneous problem. (Do not worry about convergence problems.)
Use this method to solve the inhomogeneous transport. *(2 + 4 points)*

Solution. We just compute

$$\begin{aligned}\partial_t u(x,t) &= u_t(x,t) + \int_0^t \partial_t u_s(x,t)\, ds \\ &= f(x,t) + \int_0^t L u_s(x,t)\, ds \\ &= f(x,t) + L\int_0^t u_s(x,t)\, ds \\ &= f(x,t) + Lu(x,t).\end{aligned}$$

Here we have interchanged the operator L and the integral without checking whether this is technically valid.

Suppose we have an inhomogeneous transport problem with zero initial condition.

$$\dot{u} + b \cdot \nabla u = f, \quad u(x,0) = 0.$$

Duhamel's principle tells us we should instead solve, for each s,

$$\dot{u}_s + b \cdot \nabla u_s = 0, \quad u_s(x,s) = f(x,s).$$

In the script we show how to solve the initial value problem when the initial value is at time 0. To solve a problem where the initial value is at time s, we need to do a change of coordinates $v_s(x,t') = u_s(x,t'+s)$. This gives us

$$\partial_{t'} v_s + b \cdot \nabla v_s = 0, \quad v_s(x,0) = f(x,s)$$

which has the solution $v_s(x,t') = f(x - bt', s)$. Thus $u_s(x,t) = f(x - b(t-s), s)$. Finally we get the solution

$$u(x,t) = \int_0^t f(x - b(t-s), s)\, ds.$$

This is the same as the solution we derived in Section 1.2.

12. Around and around

Consider the unit circle $C = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$. In this question we will evaluate the integral

$$\int_C x\, d\sigma$$

in two different ways, so demonstrate that it does not depend on the choice of parametrisation.

(a) In Definition 2.3 why (or under what conditions) is it enough to cover K except for a finite number of points without changing the value of the integral? *(1 bouns point)*

(b) Take $A = K = C$ in Definition 2.3. Consider the parametrisation $\Phi : (0, 2\pi) \to C$ given by $t \mapsto (\cos t, \sin t)$. Compute the integral using this parametrisation. *(2 points)*

(c) Consider upper and lower halves of the circle: $U_1 = \{(x, y) \in C \mid y > 0\}$ and $U_2 = \{(x, y) \in C \mid y < 0\}$. There are obvious parametrisations $\Phi_i : (-1, 1) \to U_i$ given by $\Phi_1(x) = (x, +\sqrt{1 - x^2})$ and $\Phi_2.(x) = (x, -\sqrt{1 - x^2})$. Compute the integral using these parametrisations. *(2 points)*

(d) (Optional) Construct a non-trivial partition of unity for the circle and compute the integral. Hint. The easiest way is to use two parametrisations similar to part (b).

(e) Compute this integral using the divergence theorem. *(2 points)*

Solution.

(a) This depends somewhat on the definition of integral that you are using. In Lebesgue integration, sets of measure zero can not contribute to the final value, and a finite collection of points is measure zero in dimensions 1 and higher. With Darboux or Riemann integrations, these are defined initially on closed sets only. They are extended to open sets, or in this case punctured neighbourhoods by taking a limit of closed sets. Continuity of the integrand is sufficient then to ensure there is no difference.

(b) Using the previous part, we know that we can integrate with the parametrisation over $U = (0, 2\pi)$, and simply ignore the point $(1, 0) \in C$ that is not covered because that does not affect the value of the integral.

We must also calculate the area-element factor. The coordinate maps $\Phi : (0, 2\pi) \subset \mathbb{R} \to \mathbb{R}^2$, so its derivative is size 2×1, namely $(-\sin t, \cos t)^T$. The factor therefore is

$$\det \begin{bmatrix} -\sin t & \cos t \end{bmatrix} \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} = \det \begin{bmatrix} 1 \end{bmatrix} = 1$$

We can now compute the integral

$$\int_0^{2\pi} \cos t \times 1 \, dt = \sin t \Big|_0^{2\pi} = 0.$$

(c) Here we have that $\Phi_1' = (1, -x(1 - x^2)^{-0.5})^T$. So

$$\int_{U_1} x \, d\sigma = \int_{-1}^{1} x \sqrt{1 + x^2(1 - x^2)^{-1}} \, dx = 0$$

(using oddness). And likewise for U_2.

(d) As hinted at, start with the parametrisation Φ defined in part (b). Now take a bump function h on $(0, 2\pi)$ such that it is identically 1 on $[\pi/2, 3\pi/2]$ and has compact support K strictly contained in the interval. Now, we need a second coordinate chart to cover the point $t = 0$, ie $(1, 0) \in C$. For this we use $\Psi : (-\pi/2, \pi/2)$ given by $\Psi(t) = (\cos t, \sin t)$. Because it has the same formula, the area-element of Ψ is also 1.

Notice that $V_1 = C \setminus \{(1, 0)\}$, $h_1 = h$, $V_2 = C \cap \{x > 0\}$, and $h_2 = 1 - h$ is a partition of unity for C, and that Φ and Ψ are coordinates for the two set respectively. Hence

$$\begin{aligned}
\int_C x \, d\sigma &= \int_0^{2\pi} h(t) \cos t \, dt + \int_{-\pi/2}^{\pi/2} (1 - h(t)) \cos t \, dt \\
&= \int_{\pi/2}^{3\pi/2} h(t) \cos t \, dt + \int_{-\pi/2}^{\pi/2} h(t) \cos t \, dt + \int_{-\pi/2}^{\pi/2} (1 - h(t)) \cos t \, dt \\
&= \int_{\pi/2}^{3\pi/2} \cos t \, dt + \int_{-\pi/2}^{\pi/2} (1 - h(t) + h(t)) \cos t \, dt \\
&= \int_{-\pi/2}^{3\pi/2} \cos t \, dt,
\end{aligned}$$

which comes out to the same calculation as in part (b). Thus we see that a partition of unity is a technical tool to divide an integral into pieces without worrying about overlaps and/or missing points, but not so practical for calculation.

(e) To apply the divergence theorem, we recognise C as the boundary of the disc $\Omega = \{x^2+y^2 \leq 1\}$, with the outward pointing normal $N = (x, y)$. (Because this it is the unit circle, this normal N is already unit length.) We now need to write the integrand x in the form $f \cdot N = (f_1, f_2) \cdot N = xf_1 + yf_2$. We see that $f = (y, 0)$ fits. The divergence of f is

$$\nabla \cdot f = \frac{\partial}{\partial x} y + \frac{\partial}{\partial y} 0 = 0.$$

Therefore

$$\int_C xy \, d\sigma = \int_{\partial\Omega} f \cdot N \, d\mu = \int_\Omega \nabla \cdot f \, d\mu = \int_\Omega 0 \, d\mu = 0.$$