Martin Schmidt
Ross Ogilvie

Introduction to Partial Differential Equations
Exercise sheet 4

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10. Solving PDEs Solve the initial value problems of the following PDEs using the method of characteristics. You may assume that g is continuously differentiable on the corresponding domain.

(a) $x_1 \partial_2 u - x_2 \partial_1 u = u$ on the domain $x_1, x_2 > 0$, with initial condition $u(x_1, 0) = g(x_1)$.
(4 points)

(b) $x_1 \partial_1 u + 2x_2 \partial_2 u + \partial_3 u = 3u$ on $x_1, x_2 \in \mathbb{R}$, $x_3 > 0$, with initial condition $u(x_1, x_2, 0) = g(x_1, x_2)$.
(4 points)

(c) $u \partial_1 u + \partial_2 u = 1$ on the domain $x_1, x_2 > 0$, with initial condition $u(x_1, x_1) = \frac{1}{2} x_1$.
(5 points)

11. Duhamel's Principle

Duhamel's principle occurs in a few places in the script. In this exercise we give the general idea and show how it applies to the transport equation. It is a method to solve an inhomogeneous PDE on $\mathbb{R}^n \times \mathbb{R}$ of the following form

$$\partial_t u - Lu = f(x,t), \quad u(x,0) = 0,$$

where L is a linear differential operator on $\mathbb{R}^n$ with constant coefficients. The idea is to instead we consider the following family of homogeneous equations

$$\partial_t u_s - Lu_s = 0, \quad u_s(x,s) = f(x,s).$$

Suppose that we can find such solutions u_s. Prove that

$$u(x,t) = \int_0^t u_s(x,t)\, ds$$

is a solution to the inhomogeneous problem. (Do not worry about convergence problems.)
Use this method to solve the inhomogeneous transport. *(2 + 4 points)*

12. Around and around

Consider the unit circle $C = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$. In this question we will evaluate the integral

$$\int_C x\, d\sigma$$

in two different ways, so demonstrate that it does not depend on the choice of parametrisation.

(a) In Definition 2.3 why (or under what conditions) is it enough to cover K except for a finite number of points without changing the value of the integral? *(1 bouns point)*

(b) Take $A = K = C$ in Definition 2.3. Consider the parametrisation $\Phi : (0, 2\pi) \to C$ given by $t \mapsto (\cos t, \sin t)$. Compute the integral using this parametrisation. *(2 points)*

(c) Consider upper and lower halves of the circle: $U_1 = \{(x,y) \in C \mid y > 0\}$ and $U_2 = \{(x,y) \in C \mid y < 0\}$. There are obvious parametrisations $\Phi_i : (-1,1) \to U_i$ given by $\Phi_1(x) = (x, +\sqrt{1-x^2})$ and $\Phi_2.(x) = (x, -\sqrt{1-x^2})$. Compute the integral using these parametrisations. *(2 points)*

(d) (Optional) Construct a non-trivial partition of unity for the circle and compute the integral. Hint. The easiest way is to use two parametrisations similar to part (b).

(e) Compute this integral using the divergence theorem. *(2 points)*