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Introduction to Partial Differential Equations
Exercise sheet 3

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7. You're not in traffic, you are traffic.

In this question we look at an equation similar to Burgers' equation that describes traffic. Let u measure the number of cars in a given distance of road, the car density. We have seen that f should be interpreted as the flux function, the number of things passing a particular point. When there are no other cars around, cars travel at the speed limit s_m. When they are bumper-to-bumper they can't move, call this density u_m.

(a) What properties do you think that f should have? Does $f(u) = s_m u \cdot (1 - u/u_m)$ have these properties? *(2 points)*

(b) Find a function f that meets your conditions, or use the f from the previous part, and write down a PDE to describe the traffic flow. *(1 point)*

(c) Find all solutions that are constant in time. *(2 points)*

(d) Consider the situation of a traffic light at $x = 0$: to the left of the traffic light, the cars are queued up at maximum density. To the right of the traffic light, the road is empty. Now, at time $t = 0$, the traffic light turns green. Give a discontinuous solution that obeys the Rankine-Hugonoit condition, as well as a continuous solution. *(6 points)*

Solution.

(a) The density flux of the cars should be the density of the cars multiplied by the speed they are travelling $f = us$. We already know that speed depends on the car density u, being zero for $u = u_m$ and s_m for $u = 0$. Assuming a linear relationship gives $s(u) = s_m(1 - u/u_m)$ and the f in the question. A more realistic relationship between density and speed would be non-linear, but probably still monotone and concave.

(b) Now, cars are a conserved quantity; have you ever seen a car vanish? Therefore it is reasonable to use the conservation PDE model. Differentiating f from the previous part gives

$$\dot{u} + s_m\left(1 - 2\frac{u}{u_m}\right)\partial_x u = 0.$$

(c) If a solution is constant in time, then it must be that $1 - u = 0$ or $\partial_x u = 0$. In either case, u must be constant. Conversely, all constant solutions solve the PDE.

(d) Choose units so that $s_m = 1$ and $u_m = 2$. The PDE is now

$$\dot{u} + (1 - u)\,\partial_x u = 0.$$

The characteristics are $x = x_0 + (1 - u_0(x_0))t$, in other words

$$\begin{cases} x = x_0 - t & \text{for } x_0 < 0 \\ x = x_0 + t & \text{for } x_0 > 0. \end{cases}$$

Physically we can explain this as there being a region $x > t$ where the first car at the light, now driving full-speed, has not yet reached and another region $x < -t$ where the traffic is still completely packed and the cars cannot move. In between cars can move with some speed, but density prevents them from moving at full speed.

This middle region is not determined by the initial conditions, so there is possibility to have many solutions. If there was a jump between maximum density and no cars, then the Rankine-Hugonoit condition would say it would have a slope of

$$\dot{y} = \frac{u_m(1 - u_m/u_m) - 0(1 - 0/u_m)}{u_m - 0} = 0.$$

The interpretation is that the lights turn green and nobody moves. This is consistent with the equations if you say that the cars at the front are also in a maximum density region.

But we prefer solutions that are as regular as possible (and also drivers who drive when the light is green, honk honk). Again, by characteristics, if $u(x,t)$ is C^1 in this region then it must be constant on lines through the origin: $x = ct$ for $c \in [-1, 1]$. It must therefore be equal to some function $g(x/t) = g(c)$ with $g(-1) = 2$ and $g(1) = 0$. The PDE then reduces to an ODE.

$$\begin{aligned} -\frac{x}{t^2} g' + (1 - g) \cdot \frac{1}{t} g' &= 0 \\ -cg' + (1 - g)g' &= 0 \\ g' \cdot (-c + 1 - g) &= 0. \end{aligned}$$

So either $g(c)$ is constant, which contradicts the endpoint conditions, or $g(c) = 1 - c$. In summary

$$u(x,t) = \begin{cases} 2 & \text{for } x < -t \\ 1 - \frac{x}{t} & \text{for } -t \le x \le t \\ 0 & \text{for } t < x \end{cases}$$

is a continuous solution.

8. Method of characteristics for an Inhomogeneous PDE Use the method of characteristics to solve the following *inhomogeneous* PDE. Note, the function u will *not* be constant along the characteristic, but its value along the characteristic will be determined by its initial value.

$$x\partial_x u + y\partial_y u = 2u$$

on the domain $x > 0, y \in \mathbb{R}$, with initial condition $u(1, y) = y$. *(5 points)*

Solution. As before, we consider a path $(x(s), y(s))$ in the domain and compute how u changes along this path:

$$\frac{du}{ds} = \partial_x u \cdot x' + \partial_y \cdot y'.$$

If we choose a path with $x' = x$ and $y' = y$, then

$$\frac{du}{ds} = x\partial_x u + y\partial_y u = 2u.$$

The characteristic is the parametric curve $x(s) = x_0 e^s$, $y(s) = y_0 e^s$. We want that the characteristic to be at the boundary $x = 1$ for $s = 0$, which means we should choose $x_0 = 1$. In non-parametric form the characteristics are the lines $y - y_0 x = 0$. These are lines which pass through the origin. The point (x_1, y_1) belongs to the characteristic with $y_0 = y_1/x_1$. Every point in the domain belongs to exactly one characteristic and every characteristic passes through the boundary condition, so there is a unique solution.

If we integrate the differential equation for u, the value of u changes along the characteristic according to $u(s) = u_0 e^{2s}$ where $u_0 = u(s = 0) = u(1, y_0)$. So to find the value of u at the point (x_1, y_1), not only do we have to find which characteristic the point belongs to, we also have to find out the value of s at that point. The value of u_0 is

$$u_0 = u(s = 0) = u(1, y_0) = y_0 = y_1/x_1.$$

The value of s at (x_1, y_1) is $e^s = x_1$, which is easily seen from the equation for $x(s)$. Therefore

$$u(x_1, y_1) = u_0(e^s)^2 = \frac{y_1}{x_1}(x_1)^2 = x_1 y_1.$$

Since this holds for any point (x_1, y_1), we should write $u(x, y) = xy$. This is the solution. Another way to explain this is that we labelled the point (x_1, y_1) with subscript 1 to distinguish it from the parametric functions $x(s), y(s)$, but now we have finished the question we can throw away the characteristics.

9. Linear Partial Differential Equations

(a) Let $b : \mathbb{R}^n \to \mathbb{R}^n$ and $c : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable functions. Then, let $x : I \to \mathbb{R}^n$ be a solution of the ordinary differential equation

$$\dot{x}(s) = b(x(s))$$

and $u : \mathbb{R}^n \to \mathbb{R}$ be a solution of the homogeneous, linear partial differential equation

$$b(x) \cdot \nabla u(x) + c(x)u(x) = 0.$$

Show that the function $z(s) := u(x(s))$ is a solution of the ordinary differential equation

$$\dot{z}(s) = -c(x(s))z(s).$$

(2 points)

(b) Consider a PDE of the form $F(\nabla u(x), u(x), x) = 0$. Suppose that F is linear in the derivatives and has continuously differentiable coefficients. That is, it can be written in the form

$$F(p, z, x) = b(z, x) \cdot p + c(z, x)$$

with b and c continuously differentiable. Show that the characteristic curves $(x(s), z(s))$ for $z(s) := u(x(s))$ can be described by ODEs that are independent of $p(s) := \nabla u(x(s))$.

. *(4 points)*

(c) With the help of the previous part, re-derive the solution of the inhomogeneous transport equation. *(3 points)*

Solution.

(a) By computation

$$\frac{d}{ds}z(s) = \nabla u(x(s)) \cdot \dot{x}(s) = \nabla u(x(s)) \cdot b(x(s)) = -c(x(s))u(x(s)) = -c(x(s))z(s).$$

Note that the previous question was of this form with $(x, y) \cdot \nabla u - 2u = 0$. Indeed, we chose our characteristics as $\dot{x} = x$ and $\dot{z} = 2z$.

(b) This part is only slightly more general than the previous part. We try to apply the method of characteristics as we have been using it to this point. We differentiate z with respect to s using the chain rule:

$$\frac{dz}{ds} = \nabla u(x(s)) \cdot \dot{x}(s) = p(s) \cdot \dot{x}(s)$$

We see that if we choose $\dot{x} = b(z, x)$ then we continue

$$= p(s) \cdot \dot{x}(s) = p \cdot b(z, x) = -c(z, x).$$

Now we have a system of ODES, namely

$$\dot{x} = b(z, x), \quad \dot{z} = -c(z, x),$$

that only involves z and x.

Let us now give some commentary on this case compared to the general case. In particular we see here how it is the fact that the PDE has a term like $b \cdot p$ that allows us to simplify the equation for z so well. In general a first order PDE does not need to have a term $b \cdot p$ and thus the equation for $\dot{z}$ will depend on p. If the equation for $\dot{z}$ depends on p, then we also need to know how p is changing along the curve. Following Section 1.5, p changes according to

$$\frac{dp}{ds} = \left(\sum_j \partial_i \partial_j u \, \dot{x}_j \right) = \mathrm{Hess}(u)\dot{x},$$

where $\mathrm{Hess}(u)$ is the matrix of second derivatives of u. The problem is that this is not an ODE in x, z, p because it depends on the second derivatives of u. The trick in the general case is to look at the total derivative of F with respect to x_i:

$$\begin{aligned} 0 &= \partial_p F \cdot \partial_i p + \partial_z F \partial_i z + \partial_i F \\ &= \partial_p F \cdot \partial_i p + \partial_z F \, p_i + \partial_i F \\ 0 &= \mathrm{Hess}(u) \nabla_p F + \partial_z F \, p + \nabla_x F. \end{aligned}$$

This equation also has a $\mathrm{Hess}(u)$ in it, but all other terms just involve x, z, p. If we suppose that the characteristic has the property that $\dot{x} = \partial_p F$, then we can use this equation to eliminate the $\mathrm{Hess}(u)$ from the $\dot{p}$ equation

$$\dot{p} = -\partial_z F \, p - \nabla F.$$

So $p(s)$ is described by an ODE and the assumption about $\dot{x}$ does not involve p.

Notice that our choice of characteristic $\dot{x} = \partial_p F$ also only depends on x, z, p. Finally,

$$\dot{z} = \nabla u \cdot \dot{x} = p \cdot \partial_p F$$

also only depends on x, z, p. So we have arrived at a system of ODEs in x, z, p.

The characteristics of the general case are actually the same as the ones in the simpler case, because $\dot{x} = \partial_p F = b$.

(c) The inhomogeneous transport equation is defined by

$$F(p, z, x) = \tilde{b} \cdot p - f(\tilde{x})$$

where $\tilde{x} = (x, t)$ and $\tilde{b} = (b, 1)$ in $\mathbb{R}^{n+1}$. From the equations we have just derived, we see that $\dot{\tilde{x}}(s) = (b, 1)$ tells us that the characteristic lines are straight lines $\tilde{x}(s) = (bs + x_0, s)$. Or in non-parametric form $x = bt + x_0$. The next ODE is $\dot{z}(s) = f(x(s), s) = f(x_0 + bs, s)$. This too can be directly integrated now

$$z(t) - z(0) = \int_0^t \dot{z}(s) \, ds = \int_0^t f(x_0 + bs, s) \, ds = \int_0^t f(x - bt + bs, s) \, ds.$$

Together with the initial condition $z(0) = u(x(0), 0) = u(x_0, 0) = g(x - bt)$ this is exactly the solution that we found previously.