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Introduction to Partial Differential Equations
Exercise sheet 3

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7. You're not in traffic, you are traffic.

In this question we look at an equation similar to Burgers' equation that describes traffic. Let u measure the number of cars in a given distance of road, the car density. We have seen that f should be interpreted as the flux function, the number of things passing a particular point. When there are no other cars around, cars travel at the speed limit s_m. When they are bumper-to-bumper they can't move, call this density u_m.

(a) What properties do you think that f should have? Does $f(u) = s_m u \cdot (1 - u/u_m)$ have these properties? *(2 points)*

(b) Find a function f that meets your conditions, or use the f from the previous part, and write down a PDE to describe the traffic flow. *(1 point)*

(c) Find all solutions that are constant in time. *(2 points)*

(d) Consider the situation of a traffic light at $x = 0$: to the left of the traffic light, the cars are queued up at maximum density. To the right of the traffic light, the road is empty. Now, at time $t = 0$, the traffic light turns green. Give a discontinuous solution that obeys the Rankine-Hugonoit condition, as well as a continuous solution. *(6 points)*

8. Method of characteristics for an Inhomogeneous PDE Use the method of characteristics to solve the following *inhomogeneous* PDE. Note, the function u will *not* be constant along the characteristic, but its value along the characteristic will be determined by its initial value.

$$x\partial_x u + y\partial_y u = 2u$$

on the domain $x > 0, y \in \mathbb{R}$, with initial condition $u(1, y) = y$. *(5 points)*

9. Linear Partial Differential Equations

(a) Let $b : \mathbb{R}^n \to \mathbb{R}^n$ and $c : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable functions. Then, let $x : I \to \mathbb{R}^n$ be a solution of the ordinary differential equation

$$\dot{x}(s) = b(x(s))$$

and $u : \mathbb{R}^n \to \mathbb{R}$ be a solution of the homogeneous, linear partial differential equation

$$b(x) \cdot \nabla u(x) + c(x)u(x) = 0.$$

Show that the function $z(s) := u(x(s))$ is a solution of the ordinary differential equation

$$\dot{z}(s) = -c(x(s))z(s).$$

(2 points)

(b) Consider a PDE of the form $F(\nabla u(x), u(x), x) = 0$. Suppose that F is linear in the derivatives and has continuously differentiable coefficients. That is, it can be written in the form

$$F(p, z, x) = b(z, x) \cdot p + c(z, x)$$

with b and c continuously differentiable. Show that the characteristic curves $(x(s), z(s))$ for $z(s) := u(x(s))$ can be described by ODEs that are independent of $p(s) := \nabla u(x(s))$.

. *(4 points)*

(c) With the help of the previous part, re-derive the solution of the inhomogeneous transport equation. *(3 points)*