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Introduction to Partial Differential Equations
Exercise sheet 2

13th September, 2022

4. Inhomogeneous Transport Equation. First order partial differential equations share many things in common with first order ordinary differential equations (ODEs). Consider the linear inhomogeneous equation

$$\frac{du}{dt} = f(t).$$

(a) Find a solution $u : \mathbb{R} \to \mathbb{R}$ to this equation. *(1 point)*

(b) For any initial value $c \in \mathbb{R}$, show that there is a unique solution with $u(0) = c$. *(2 points)*

We consider now the inhomogeneous transport equation

$$\partial_t u + b \cdot \nabla u = f$$

with initial value given by a function $g(x)$, namely $u(x, 0) = g(x)$. It had an explicit solution

$$u(x, t) = g(x - tb) + \int_0^t f(x + (s - t)b, s)\, ds.$$

(c) Show that the integral term itself solves the inhomogeneous transport equation. What initial value problem does it solve? *(3 points)*

(d) Prove that the solution to the initial value problem is unique. (You may assume that the solution to the homogeneous version is unique, if you haven't seen the lecture/read the script.) *(2 points)*

5. Royale with Cheese

Recall Burgers' equation from Example 1.6 of the lecture script:

$$\dot{u} + u\partial_x u = 0,$$

for $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. In this question we will apply the method of characteristics to solve this equation for the initial condition $u_0(x) = x$.

(a) According to Theorem 1.5, there is a unique C^1 solution to this initial value problem, at least when t is small. For how long does the theorem guarantee that the solution exists uniquely? *(1 point)*

(b) Suppose that u is a solution to this equation and suppose that $(x(s), t(s))$ is a path in the domain of u. What is the s derivative of u along this path? What constraints should we place on the derivatives of x and t? *(2 points)*

(c) On an (x, t)-plane, draw the characteristics and describe the behaviour of this solution. *(2 points)*

(d) Finally, derive the following solution to the initial value problem: *(2 points)*

$$u(x, t) = \frac{x}{1 + t}.$$

(e) Is this solution well-defined? Check by substitution that actually solves the initial value problem. *(2 points)*

(f) Why is the method of characteristics well-suited to solving first order PDEs that are linear in the derivatives? *(1 point)*

6. It's just a jump to the left

In this question we explore some other solutions to the initial value problem from Example 1.7. As we saw, for small t the method of characteristics gives a unique solution

$$u_{t<1}(x,t) = \begin{cases} 1 & \text{for } x < t \\ \frac{x-1}{t-1} & \text{for } t \leq x < 1 \\ 0 & \text{for } 1 \leq x. \end{cases}$$

(a) (Optional) Derive this solution for yourself, for extra practice.

After $t = 1$, the characteristics begin to cross and so the method cannot assign which value u should have at a point (x,t). However, we could still arbitrarily decide to choose a value of one characteristic. Consider therefore

$$v(x,t) = \begin{cases} u_{t<1} & \text{for } t < 1 \\ 1 & \text{for } x < t \\ 0 & \text{for } t \leq x \end{cases}$$

(b) Draw the corresponding characteristics diagram in the (x,t)-plane for this function. *(2 points)*

(c) Describe the graph of discontinuities $y(t)$. Compute the Rankine-Hugonoit condition for v. *(3 points)*

(d) How much mass (i.e. the integral of v over x) is being lost in the system described by v for $t > 1$? *(2 points)*