In class problems for Exercise Sheet 4

(1) Variant of 10(c)

$u\,\partial_1 u + \partial_2 u = -x_1$ with $u(x_1, 0) = 0$

Characteristics

$x_1 = x_{10}\cos s + z_0 \sin s$

$x_2 = s + x_{20}$

$z = -x_{10}\cos s + z_0 \cos s$

These are spirals in (x_1, x_2, z)-space.

Initial conds: $x_{20} = 0$, $z_0 = 0$

Solution $u(x_1, x_2) = -x_1 \tan x_2$.

(2) Example with p in the characteristic equations

$\partial_t u + (\partial_x u)^2 = t$, $u(0,x) = g(x)$ Let $x_1 = t$ $x_2 = x$

Characteristics. $F(p_1, p_2, z, x_1, x_2) = p_1 + p_2^2 - x_1$

$x_1' = \cancel{2p_2}\ 1$ $x_2' = 2p_2$

$z' = 2p_2^2 + p_1$

$p_1' = 1$ $p_2' = 0$

Solving

$p_1 = s + p_{10}$ $p_2 = p_{20}$

$x_1 = s + x_{10}$

$x_2' = 2p_{20} \Rightarrow x_2 = 2p_{20}s + x_{20}$

$z' = s + (p_{10} + 2p_{20}^2) \Rightarrow z = \frac{1}{2}s^2 + (p_{10} + 2p_{20}^2)s + z_0.$

Initial Conds: $x_{10} = 0$ $z_0 = g(x_2)$.

How to figure out p_{10}, p_{20}?

$$p_{20} = \frac{\partial u}{\partial x_2}(0, x_{20}) \xrightarrow{\text{chain rule}} = \frac{\partial}{\partial x_2}\left[u(0, x_{20})\right] = \frac{\partial g}{\partial x_2}(x_{20})$$

And $p_{10} + p_{20}^2 - x_{10} = 0$ gives p_{10}.

For $g=0$ you get

$x_{10}=0,\ z_0=0,\ p_{20}=0,\ p_{10}=0-0^2=0$

Hence $x_1=s,\ x_2=x_{20},\ z=\frac{1}{2}s^2,\ p_1=s,\ p_2=0$.

$\Rightarrow u=\frac{1}{2}x_1^{\,2}=\frac{1}{2}t^2$.

③ Eikonal Equation – Used in optics, distance function to boundary.

$$|\nabla u|=\frac{1}{\text{speed}}.$$

Challenge: $(\partial_x u)^2+(\partial_y u)^2=1,\quad u(0,x_2)=\frac{1}{2}x_2$

$F=p_1^2+p_2^2-1=0.$

1. Characteristics:

$x_1'=2p_1,\quad x_2'=2p_2,\quad p_1'=0,\quad p_2'=0,$

$z'=2p_1^2+2p_2^2=2(p_1^2+p_2^2)=2$

2. Solve:

$p_1=p_{10},\ p_2=p_{20},\ z=2s+z_0,\ x_1=2p_{10}s+x_{10},\ x_2=2p_{20}s+x_{20}$

3. Initial Conds:

$x_{10}=0,\quad z_0=\frac{1}{2}x_{20},\quad p_{20}=\frac{1}{2},\quad p_{10}=\pm\sqrt{1-\frac{1}{4}}=\pm\frac{\sqrt{3}}{2}$

4. Get u:

$x_1=\pm\sqrt{3}\,s$

$x_{20}=x_2\mp\frac{x_1}{\sqrt{3}}$

$$u=2\left(\pm\frac{x_1}{\sqrt{3}}\right)+\frac{1}{2}\left(x_2\mp\frac{x_1}{\sqrt{3}}\right)=\frac{1}{2}x_2\pm x_1\left(\frac{2}{\sqrt{3}}-\frac{1}{2\sqrt{3}}\right)$$

$$=\frac{1}{2}x_2\pm\frac{\sqrt{3}}{2}x_1$$

For prison yard, $x_1>0$, and increasing x_1 increases time.

So $u=\frac{1}{2}x_2+\frac{\sqrt{3}}{2}x_1$