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Analysis III
13. Exercise: Orientation

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77. Orientable hypersurfaces defined by an equation.

Let $f : \mathbb{R}^n \to \mathbb{R}$ a smooth map, $q \in \mathbb{R}$ a point in its range so $X := f^{-1}(\{q\}) \neq \varnothing$, and f is submersive at all points $x \in X$. Show that X is an $(n-1)$-dimensional orientable submanifold.

Hint. Let ω be the volume form on $\mathbb{R}^n$ and F the gradient field of f (i.e $T_x(f)(v) = F(x) \cdot v$). Investigate $i_F\omega|X$, defined in Definition 3.11.

Solution. Because f is a submersion, we know that X is an $(n-1)$-dimensional submanifold. The gradient field $F = \nabla f$ is a vector field on $\mathbb{R}^n$, so it makes sense to contract the volume form with it. $\iota_F\omega$ is a $(n-1)$-form, so we should try show that this form is non-vanishing on X, so that Theorem 3.17(iv) applies.

Choose any point $x \in X$ and take a basis v_i of T_xX. Then $\{F(x), v_1, \ldots, v_{n-1}\}$ is a basis of $T_x\mathbb{R}^n$ because F is perpendicular to the vectors v_i and does not vanishing at x because f is a submersion. Hence

$$\langle \iota_F\omega, v_1 \otimes \cdots \otimes v_{n-1}\rangle = \langle \omega, F \otimes v_1 \otimes \cdots \otimes v_{n-1}\rangle \neq 0$$

shows that $\iota_F\omega|X$ is non-vanishing.

In Class Exercises

78. Orientable manifolds.

(a) Show that the n-dimensional sphere $\mathbb{S}^n$ is orientable by finding an oriented atlas.

Hint. For the sphere, consider the atlas that uses stereographic projection. An extra trick is also needed.

(b) Show that the Möbius band is not orientable.

Hint. This is difficult. Good luck.

(c) Let X and Y be orientable manifolds. Show that the Cartesian product $X \times Y$ is also orientable.

(d) Let X be a manifold. Show that every coordinate neighbourhood of X is orientable. More precisely, let (U, ϕ) be a chart of X with $\phi = (\phi_1, \ldots, \phi_n) : U \to \mathbb{R}^n$, and show that $\mathrm{d}\phi_1 \wedge \ldots \wedge \mathrm{d}\phi_n$ is a non-vanishing n-form on U.

(e) Prove that the tangent bundle of any manifold is orientable.

Solution.

(a) As we have seen many times, we can cover the sphere with two charts using stereographic projection. The transistion function between these two charts was already computed in Example 1.18(iii) as $y \mapsto z := \|y\|^{-2}y$ for $y \in \mathbb{R}^n \setminus \{0\}$. We have that

$$\frac{\partial z_j}{\partial y_i} = \begin{cases} \frac{\|y\|^2 - 2y_i^2}{\|y\|^4} & \text{for } i = j \\ \frac{-2y_i y_j}{\|y\|^4} & \text{for } i \neq j. \end{cases}$$

Thus we need to find the sign of the matrix $\|y\|^2 I - (2y_i y_j)_{i,j}$. Recall that the determinant is the product of the eigenvalues of a matrix. The eigenvalues of the matrix $bI - A$ are related to those of A because

$$(bI - A)v = \lambda v \Leftrightarrow Av = (b - \lambda)v.$$

Thus we need to calculate the eigenvalues of $(2y_i y_j)_{i,j}$. But we recognise that this is the product $2yy^T$ for y a column vector (notice this is not the familiar order used to write the dot product). We see immediately that y is itself an eigenvector, since

$$(2yy^T)y = 2y(y^T y) = 2\|y\|^2 y.$$

On the other hand, if v is perpendicular to y, then the same computation shows that v is a null vector of A. This gives us a basis of eigenvectors of A and all of the eigenvalues. Therefore the eigenvalues of $\|y\|^2 I - (2y_i y_j)_{i,j}$ are $\|y\|^2 - 2\|y\|^2 = -\|y\|^2$ and $n-1$ copies of $\|y\|^2$. This shows that transition matrix has everywhere *negative* determinant and the atlas of the two stereographic projections is not an oriented atlas.

However, we can easily make an oriented atlas now. Let ϕ_N, ϕ_S be stereographic projection from the north and south pole respectively. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be map $A(x_1, \ldots, x_n) = (-x_1, x_2, \ldots, x_n)$ which reflects in the first coordinate. Consider the atlas $\{\phi_N, A \circ \phi_S\}$. There is only one transition to consider here, and the determinant of the derivative of the transition is, by the chain rule,

$$\det(A \circ \phi_S \circ \phi_N^{-1})' = \det A' \det(\phi_S \circ \phi_N^{-1}) = -\det(\phi_S \circ \phi_N^{-1}) > 0.$$

Thus this is an oriented atlas, and shows $\mathbb{S}^n$ is oriented.

There is also a more abstract way to make this argument that avoids the computation of the determinant: Note that the Jacobian is everywhere full-rank because it has no kernel (or that the transition function is diffeomorphism). Therefore its determinant is non-zero and so must be a single sign on $\mathbb{R}^n \setminus \{0\}$ (this set is connected and the determinant is continuous, so the image must be connected, hence it has a single sign). If the sign is positive, we have an oriented atlas. If the sign is everywhere negative, compose one of the charts with the reflection A of $\mathbb{R}^n$. The result is an oriented atlas. In either case, $\mathbb{S}^n$ must be orientable.

(b) To prove something is non-orientable, we must show that there does not exist any oriented atlas for the manifold; as we have seen in the previous part it is not enough to say that the usual atlas is not oriented. For this reason, will be prove that the Möbius band is non-orientable by contradiction.

But first, let us recall the accoutrements for the Möbius band $M \subset \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$. Let the standard atlas for $\mathbb{R}/\mathbb{Z}$ be $\{(U_x, \phi_x)\}_{x\in\mathbb{R}}$. We have seen that this bundle trivialises over a cover with two sets, for example U_0 and $U_{0.5}$. Using the function $\ell(x) = (\cos \pi x, \sin \pi x) \in L(x)$, the trivialisations are $\Phi_x(t, [y]) = t\ell(\phi_x([y])) = t\ell(y)$. We understand how to use trivialisations to construct an adapted chart for the bundle. This gives us an atlas for M, namely the two charts

$$\tilde{\phi}_0 : \pi^{-1}[U_0] \to \mathbb{R} \times (-0.5, 0.5) \qquad \tilde{\phi}_{0.5} : \pi^{-1}[U_{0.5}] \to \mathbb{R} \times (0, 1)$$

$$\tilde{\phi}_0(t\ell(y)) = (t, y) \qquad \tilde{\phi}_{0.5}(t\ell(y)) = (t, y).$$

This is not an oriented atlas, as we can compute. For points with $y \in (0, 0.5)$ we have

$$\tilde{\phi}_{0.5} \circ \tilde{\phi}_0^{-1}(t, y) = \tilde{\phi}_{0.5}(t\ell(y)) = (t, y).$$

The Jacobian of the change of coordinates is therefore the identity matrix and the sign of the determinant is 1. Whereas for $y \in (-0.5, 0)$ we have

$$\tilde{\phi}_{0.5} \circ \tilde{\phi}_0^{-1}(t, y) = \tilde{\phi}_{0.5}(t\ell(y)) = \tilde{\phi}_{0.5}(-t\ell(y + 1)) = (-t, y + 1).$$

Now the sign of the determinant of the Jacobian of the change of coordinates is -1.

With this prepared, we can now give the argument. Suppose that there is an oriented atlas $\{(V_i, \psi_i)\}$ of the Möbius band M. Consider the function $s_0(m) := \operatorname{sign}\det J_{\tilde{\phi}_0(m)}(\psi_i \circ \tilde{\phi}_0^{-1})$ from $\pi^{-1}[U_0]$ to $\{1, -1\}$. This is independent of the chart ψ_i because

$$\begin{aligned} \operatorname{sign}\det J(\psi_i \circ \tilde{\phi}_0^{-1}) &= \operatorname{sign}\Big(\det J(\psi_i \circ \psi_j^{-1} \circ \psi_j \circ \tilde{\phi}_0^{-1})\Big) \\ &= \operatorname{sign}\det J(\psi_i \circ \psi_j^{-1}) \operatorname{sign}\det J(\psi_j \circ \tilde{\phi}_0^{-1}) \\ &= \operatorname{sign}\det J(\psi_j \circ \tilde{\phi}_0^{-1}). \end{aligned}$$

Since the determinant is non-zero (these are coordinate charts), this is a continuous function. Further, $\pi^{-1}[U_0]$ is connected, so in fact it is a constant function. By identical argument, the function $s_{0.5}(m) := \operatorname{sign}\det J_{\tilde{\phi}_{0.5}^{-1}(m)}(\psi_i \circ \tilde{\phi}_{0.5}^{-1})$ from $\pi^{-1}[U_{0.5}]$ to $\{1, -1\}$ is also constant.

But this leads to a contradiction. At $m = (0, [0.25])$:

$$\begin{aligned} s_0(m) &= \operatorname{sign}\det J_{\tilde{\phi}_0(m)}(\psi_i \circ \tilde{\phi}_0^{-1}) \\ &= \operatorname{sign}\det J_{\tilde{\phi}_{0.5}(m)}(\psi_i \circ \tilde{\phi}_{0.5}^{-1}) \operatorname{sign}\det J_{\tilde{\phi}_{0.5}(m)}(\tilde{\phi}_{0.5} \circ \tilde{\phi}_0^{-1}) \\ &= s_{0.5}(m) \times 1. \end{aligned}$$

On the other hand, at $m = (0, [0.75])$:

$$\begin{aligned} s_0(m) &= \operatorname{sign}\det J_{\tilde\phi_0(m)}(\psi_i \circ \tilde\phi_0^{-1}) \\ &= \operatorname{sign}\det J_{\tilde\phi_{0.5}(m)}(\psi_i \circ \tilde\phi_{0.5}^{-1}) \operatorname{sign}\det J_{\tilde\phi_{0.5}(m)}(\tilde\phi_{0.5} \circ \tilde\phi_0^{-1}) \\ &= s_{0.5}(m) \times -1. \end{aligned}$$

This is impossible for constants $s_0, s_{0.5} \in \{1, -1\}$.

(c) Choose oriented atlases $\{\phi_i\}$ for X and $\{\psi_j\}$ for Y. Then the standard atlas $\{\phi_i \times \psi_j\}$ for $X \times Y$ is oriented because

$$\operatorname{sign}\det((\phi_i \times \psi_j) \circ (\phi_k \times \psi_l)^{-1})' = \operatorname{sign} \begin{vmatrix} (\phi_i \circ \phi_k^{-1})' & 0 \\ 0 & (\psi_j \circ \psi_l^{-1})' \end{vmatrix} = 1 \cdot 1 = 1.$$

(d) Let E_k be the coordinate vector fields on U. We have already seen that this n-form is 1 at every point when applied to $E_1 \otimes \cdots \otimes E_n$. Therefore it is non-vanishing.

(e) Again, there is a standard atlas on the tangent bundle TX that arises from any atlas $\{(U_i, \phi_i)\}$ on X given by the tangent map, namely

$$T(\phi_i) : \pi^{-1}[U_i] \to \mathbb{R}^n \times \phi[U_i], \quad (v, x) \mapsto (T_x(\phi_i)v, \phi_i(x)).$$

We can compute the Jacobian of the transition function $T(\phi_i \circ \phi_j^{-1})$ in block form, for $w \in \mathbb{R}^n$ and $y = \phi_j(x)$:

$$\Big(T(\phi_i) \circ T(\phi_j)\Big)'(w, y) = \begin{pmatrix} T_y(\phi_i \circ \phi_j^{-1})w & \frac{\partial}{\partial y_k} T_y(\phi_i \circ \phi_j^{-1})(w, y) \\ 0 & (\phi_i \circ \phi_j^{-1})'(w, y) \end{pmatrix}.$$

The first block is so because the derivative of a linear map is the same linear map. The off diagonal block is difficult to compute, but not needed to find the determinant. Observe finally that the two diagonal blocks are actually two different notations for the same thing. Therefore the determinant is a square, and hence always positive.

79. Integration on $\mathbb{R}$.

Consider the following integration by substitution with $f(x) = x^2$:

$$\int_{-1}^{2} 2xe^{-x^2}\, dx = \int_{-1}^{2} e^{f(x)}\, f'(x)dx = \int_{f(-1)}^{f(2)} e^u\, du = \int_1^4 e^u\, du.$$

This seems very similar to Corollary 3.22, except that f is not a diffeomorphism! Also consider the substitution with $v = -x$

$$\int_{-1}^{2} e^{-x}\, dx = \int_1^{-2} e^u\, (-du) = \int_{-2}^{1} e^u\, du.$$

The aim of this exercise is to understand why this all works for $\mathbb{R}$.

First, extend Corollary 3.22 for orientation-reversing diffeomorphisms.

Second, define

$$\int_a^b g(x)\ dx = \operatorname{sign}(b-a)\int_{\mathbb{R}} \chi_{[a,b]}(x)g(x)\ dx.$$

Show that it obeys the rule $\int_a^b + \int_b^c = \int_a^c$.

Third, let $f : [a,b] \to \mathbb{R}$ be a continuously differentiable function. The set $(f')^{-1}[\{0\}]$ is closed, and therefore the countable union of closed intervals J_i (single points are considered as the interval $[x,x]$). Taking the endpoints of these intervals gives a partition $\{t_i\}$ of $[a,b]$ such that on each interval (t_i, t_{i+1}) the function f' is either non-zero or identically zero. Prove

$$\int_a^b g(f(x))f'(x)\ dx = \int_{f(a)}^{f(b)} g(u)\ du.$$

Solution. First, if $f : X \to Y$ is orientation reversing, then it is orientation preserving if Y is given the opposite orientation. This negates all integrals on Y. Therefore

$$\int_{f[A]} \omega = -\int_A f^*\omega.$$

Second, there are several cases to consider. We will demonstrate one case: $a \le c \le b$. Then

$$\int_a^b g + \int_b^c g = \int_{\mathbb{R}} (\chi_{[a,b]} - \chi_{[c,b]})g = \int_{\mathbb{R}} \chi_{[a,c)} g = \int_a^c g,$$

since $\chi_{[a,c)}$ and $\chi_{[a,c]}$ are almost everywhere equal. The other cases are very similar.

Thirdly, decompose the integral over $[a,b]$ according to the partition, discarding the endpoints as they are sets of measure zero

$$\int_a^b g(f(x))f'(x)\ dx = \sum_i \int_{(t_i,t_{i+1})} g(f(x))f'(x)\ dx.$$

If f' is zero on (t_i, t_{i+1}) then $f(t_i) = f(t_{i+1})$ and the integral on this interval is zero. Therefore

$$\int_{(t_i,t_{i+1})} g(f(x))f'(x)\ dx = 0 = \int_{\emptyset} g(u)\ du = \int_{f(t_i)}^{f(t_{i+1})} g(u)\ du.$$

If $f' > 0$ on (t_i, t_{i+1}) then it is an orientation preserving C^1-diffeomorphism and Corollary 3.22 applies:

$$\int_{(t_i,t_{i+1})} g(f(x))f'(x)\ dx = \int_{f(t_i)}^{f(t_{i+1})} g(u)\ du.$$

since $f^*du = f'(x)\ dx$. If $f' < 0$ then f is orientation reversing, $f(t_{i+1}) < f(t_i)$, and

$$\int_{(t_i,t_{i+1})} g(f(x))f'(x)\ dx = -\int_{[f(t_i),f(t_{i+1})]} g(u)\ du = \int_{f(t_i)}^{f(t_{i+1})} g(u)\ du.$$

Putting together all three cases, we get

$$\sum_i \int_{t_i}^{t_{i+1}} g(f(x))f'(x)\ dx = \sum_i \int_{f(t_i)}^{f(t_{i+1})} g(u)\ du = \int_{f(a)}^{f(b)} g(u)\ du.$$

We might wonder whether this can be generalised. We can restrict to $\det Jf \neq 0$ without changing the value of either integral by Sard's theorem. But the next step for $\mathbb{R}$ was the fact that $f' > 0$ implies that f is injective and therefore we can apply Corollary 3.22. This inference is not true generally. Indeed it is not even true for the circle, which is exactly the problem in Beispiel 3.23.