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Analysis III
12. Exercise: Integration

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Preparation Exercises

70. The pullback of differential forms.

(a) Let X, Y be manifolds of dimension n and $f : X \to Y$ a smooth map. Further take the standard local set-up of charts $\phi = (\phi_1, \ldots, \phi_n) : U \to \mathbb{R}^n$ and $\psi = (\psi_1, \ldots, \psi_n) : V \to \mathbb{R}^n$ on open sets $U \subset X$ and $V \subset Y$ with $f(U) \subset V$.

Show the following local formula for the pullback holds for every smooth function $g \in \mathcal{C}^\infty(V, \mathbb{R})$:

$$f^*(g\,\mathrm{d}\psi_1 \wedge \cdots \wedge \mathrm{d}\psi_n) = (g \circ f) \cdot \det\left(\frac{\partial(\psi_j \circ f \circ \phi^{-1})}{\partial x_i}\right) \cdot \mathrm{d}\phi_1 \wedge \cdots \wedge \mathrm{d}\phi_n\,.$$

Hint. Make use of the determinant formula for the evaluation of forms $\langle A_1 \wedge \cdots \wedge A_p\,,\, v_1 \otimes \ldots \otimes v_p\rangle = \det\big(A_i(v_j)\big)_{i,j}$, from page 71 of the script.

(b) Consider the *canonical volume form* on $\mathbb{R}^3$, namely $\omega := \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ and *spherical coordinates*

$$f : \mathbb{R}_+ \times [0, 2\pi) \times [0, \pi] \to \mathbb{R}^3,\ (r, \vartheta, \varphi) \mapsto \big(r\,\cos(\vartheta)\,\cos(\varphi)\,,\, r\,\cos(\vartheta)\,\sin(\varphi)\,,\, r\,\sin(\vartheta)\big).$$

Compute "ω in spherical coordinates", by which we mean the pullback $f^*\omega$.

Solution.

(a) A special case of this is used in the proof of Theorem 3.17 to change between n-forms coming from two sets of coordinates on the same manifold. The operation itself is defined in Theorem 3.12(iii) and Theorem 3.5.

In particular, because we know from Theorem 3.12(iv) that pulling back is an exterior algebra homomorphism, we can compute the effect on each part and then recombine them. The pullback of a smooth function is just pre-composition: $f^*g = g \circ f$. Moreover, by Theorem 3.15(ii) pullbacks commute with exterior derivatives, so

$$f^*d\psi_j = d(f^*\psi_i) = d(\psi_j \circ f) = \sum_i \frac{\partial(\psi_j \circ f \circ \phi^{-1})}{\partial x_i}\, d\phi_i$$

using the formula for exterior derivative in terms of coordinate charts on page 75. We have one such 1-form for each $d\psi_j$, and clearly it would be a pain to try to expand out the exterior product directly. This is where we follow the hint. Let E_k be coordinate vector fields on U, that is $d\phi_i(E_k) = \delta_{i,k}$. Then

$$\begin{aligned}
\langle f^* d\psi_1 \wedge \cdots \wedge f^* d\psi_n, E_1 \otimes \cdots \otimes E_n \rangle &= \det\Big(\sum_i \frac{\partial(\psi_j \circ f \circ \phi^{-1})}{\partial x_i}\, d\phi_i(E_k)\Big)_{j,k} \\
&= \det\Big(\sum_i \frac{\partial(\psi_j \circ f \circ \phi^{-1})}{\partial x_i}\, \delta_{i,k}\Big)_{j,k} \\
&= \det\Big(\frac{\partial(\psi_j \circ f \circ \phi^{-1})}{\partial x_i}\Big)_{j,i}
\end{aligned}$$

$$\begin{aligned}
\Big\langle \det\Big(\frac{\partial(\psi_j \circ f \circ \phi^{-1})}{\partial x_i}\Big)_{i,j} d\phi_1 \wedge \cdots \wedge d\phi_n, E_1 \otimes \cdots \otimes E_n \Big\rangle \\
&= \det\Big(\frac{\partial(\psi_j \circ f \circ \phi^{-1})}{\partial x_i}\Big)_{i,j} \det\Big(d\phi_i(E_k)\Big)_{i,k} \\
&= \det\Big(\frac{\partial(\psi_j \circ f \circ \phi^{-1})}{\partial x_i}\Big)_{i,j}
\end{aligned}$$

Any other pure n-form in increasing order must repeat vectors, so for the first form leads to a repeated column (and so the determinant is zero) and for the second form leads to a column of zeroes. This shows that the two differential forms act identically, and so are equal.

(b) Here we interpret the interior of the domain of f as a manifold X and let $Y = \mathbb{R}^3$. In particular $X = (0, \infty) \times (0, 2\pi) \times (0, \pi)$ is an open subset of $\mathbb{R}^3$ so we use the identity function as the chart ϕ, but label the components r, ϑ, φ, whereas on $Y = \mathbb{R}^3$ we also use the identity function as a chart but label the coordinates with the usual x, y, z. Therefore we have

$$(x, y, z) = \psi \circ f \circ \phi^{-1}(r, \vartheta, \varphi) = f(r, \vartheta, \varphi).$$

The determinant that we need to compute is nothing other than the determinant of the Jacobian matrix:

$$\begin{vmatrix} \cos(\vartheta)\cos(\varphi) & -r\sin(\vartheta)\cos(\varphi) & -r\cos(\vartheta)\sin(\varphi) \\ \cos(\vartheta)\sin(\varphi) & -r\sin(\vartheta)\sin(\varphi) & r\cos(\vartheta)\cos(\varphi) \\ \sin(\vartheta) & r\cos(\vartheta) & 0 \end{vmatrix}$$

$$= \sin(\vartheta) \begin{vmatrix} -r\sin(\vartheta)\cos(\varphi) & -r\cos(\vartheta)\sin(\varphi) \\ -r\sin(\vartheta)\sin(\varphi) & r\cos(\vartheta)\cos(\varphi) \end{vmatrix} - r\cos(\vartheta) \begin{vmatrix} \cos(\vartheta)\cos(\varphi) & -r\cos(\vartheta)\sin(\varphi) \\ \cos(\vartheta)\sin(\varphi) & r\cos(\vartheta)\cos(\varphi) \end{vmatrix}$$

$$= r^2 \sin^2(\vartheta)\cos(\vartheta) \begin{vmatrix} \cos(\varphi) & \sin(\varphi) \\ \sin(\varphi) & -\cos(\varphi) \end{vmatrix} - r^2\cos^3(\vartheta) \begin{vmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{vmatrix}$$

$$= -r^2\cos(\vartheta)$$

71. Integration on the unit circle.

Let ω be a 1-form on the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$ and

$$f : \mathbb{R} \to \mathbb{S}^1,\ t \mapsto (\cos t, \sin t)$$

a paramterisation.

(a) Use the exercise "Null sets of manifolds" and Corollary 3.22 to calculate

$$\int_{\mathbb{S}^1} y\ dx$$

(b) Prove Stokes' theorem for $\mathbb{S}^1$. Actually, show the stronger result that ω is exact if and only if

$$\int_{\mathbb{S}^1} \omega = 0.$$

($\mathbb{S}^1$ is a manifold whose boundary is empty, so the right side of Stokes' theorem is zero.)

Solution.

(a) We know that an integral is not changed by the exclusion of a null set. Thus

$$\int_{\mathbb{S}^1} \omega = \int_{\mathbb{S}^1 \setminus \{(1,0)\}} \omega = \int_{(0,2\pi)} f^*\omega = \int_{[0,2\pi]} f^*\omega$$

using Corollary 3.22 in the middle step, because f is a diffeomorphisms between $\mathbb{S}^1 \setminus \{(1,0)\}$ and $(0, 2\pi)$.

Hence

$$\begin{aligned}\int_{\mathbb{S}^1} y\ dx + x\ dy &= \int_0^{2\pi} \sin t(-\sin t\ dt) \\ &= \int_0^{2\pi} -\sin^2 t\ dt = -\pi.\end{aligned}$$

(b) Suppose that $\omega = dg$ for a smooth function $g : \mathbb{S}^1 \to \mathbb{R}$. Then we have that

$$\int_{\mathbb{S}^1} dg = \int_{[0,2\pi]} d(g \circ f) = g(f(2\pi)) - g(f(0)) = g(1,0) - g(1,0) = 0.$$

This shows Stokes' theorem in this case.

For the stronger statement, which also has the converse, suppose that the integral of ω is 0 over the circle. Define the real function $G : \mathbb{R} \to \mathbb{R}$ by

$$G(t) = \int_0^t f^*\omega.$$

This function is in fact periodic with period 2π because f is and

$$G(t+2\pi) = \int_0^{2\pi} f^*\omega + \int_{2\pi}^{t+2\pi} f^*\omega = 0 + \int_0^t f^*\omega = G(t).$$

Thus G defines a function g on the circle. Every point of the circle has a neighbourhood where f restricts to give a coordinate chart. In this coordinate chart, we see that $G = g \circ f = f^*g$. Since $dG = G'\,dt = f^*\omega$, it follows that $dg = \omega$. This shows that ω is exact as required.

In Class Exercises

72. Null sets of manifolds.

Let M be an oriented manifold and Z a closed subset. Hence $M \setminus Z$ is also a manifold. We call Z a null set if for every coordinate chart $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ the set $\phi_\alpha[Z \cap U_\alpha]$ is a null set of $\mathbb{R}^n$. Prove that

$$\int_A \omega = \int_{A\setminus Z} \omega$$

Solution. First, a technical point: The set $A \setminus Z$ might not be compact so Definition 3.21 might not apply. But we see that the reason that A should be compact is so the sum is finite and guaranteed to exist. In this situation we know that $A \setminus Z$ can be covered by finitely many charts, so this concern about well-definition is moot.

Because the charts are bijections, $\phi_\alpha[(A \setminus Z) \cap U_\alpha] = \phi_\alpha[A \cap U_\alpha] \setminus \phi_\alpha[Z \cap U_\alpha]$. Then

$$\begin{aligned}
\int_{A\setminus Z} \omega &= \sum_m \int_{\phi_m[(A\setminus Z)\cap U_m]} \phi_m^*(f_m\omega) \\
&= \sum_m \int_{\phi_m[A\cap U_m]} \phi_m^*(f_m\omega) - \sum_m \int_{\phi_m[Z\cap U_m]} \phi_m^*(f_m\omega) \\
&= \sum_m \int_{\phi_m[A\cap U_m]} \phi_m^*(f_m\omega) - 0 \\
&= \int_A \omega.
\end{aligned}$$

73. The divergence theorem (aka Gauss' theorem).

Let $X \subset \mathbb{R}^n$ be a compact subset of $\mathbb{R}^n$ with $\overline{X^0} = X$ that is an n-dimensional manifold with boundary. It is know that X must be orientable and that $\omega := \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ is a volume form on X. Further, let a smooth $(n-1)$-form η on X be given.

(a) Show that there is a unique vector field $F \in \mathrm{Vec}^\infty(X)$ with $\eta = i_F\omega$.

(b) Write $F = (F_1, \ldots, F_n)$ for functions $F_1, \ldots, F_n \in C^\infty(X, \mathbb{R})$. Define the divergence operator $\mathrm{div}(F) \in C^\infty(X, \mathbb{R})$ as

$$\mathrm{div}(F) := \sum_{k=1}^{n} \frac{\partial F_k}{\partial x_k}.$$

Prove the following connection between the divergence operator and the exterior derivative:

$$\mathrm{d}(i_F\omega) = \mathrm{div}(F) \cdot \omega.$$

(c) Prove the divergence theorem:

$$\int_{\partial X} \eta = \int_X \mathrm{div}(F) \cdot \omega.$$

Solution.

(a) We know that we can write $\eta = \sum \eta_i dx_1 \wedge \ldots \widehat{dx_i} \cdots \wedge dx_n$. Consider then the vector field $F := (\eta_1, \ldots, (-1)^{n-1}\eta_n)$ and in particular how $\iota_F\omega$ acts on $E_1 \otimes \widehat{E_i} \otimes E_n$:

$$\begin{aligned}
\langle \iota_F\omega, E_1 \otimes \widehat{E_i} \otimes E_n\rangle &= \langle \omega, F \otimes E_1 \otimes \widehat{E_i} \otimes E_n\rangle \\
&= \sum_j \langle \omega, (-1)^{j-1}\eta_j E_j \otimes E_1 \otimes \widehat{E_i} \otimes E_n\rangle \\
&= \sum_j (-1)^{j-1}\eta_j \det(dx_k(v_l))_{k,l} \\
&= (-1)^{i-1}\eta_i \det(dx_k(v_l))_{k,l} \\
&= (-1)^{i-1}\eta_i \cdot (-1)^{i-1},
\end{aligned}$$

because the only determinant that does not have a repeated column is the one where $j = i$. For that matrix, you then have to do $j-1$ column swaps to make it the identity matrix. This shows that $\iota_F\omega$ acts identically to η.

(b) We can apply part (a) in reverse, so that $\iota_F\omega = \sum_i (-1)^{i-1} F_i dx_1 \wedge \ldots \widehat{dx_i} \cdots \wedge dx_n$.

Now we apply the exterior derivative

$$
\begin{aligned}
d(\iota_F\omega) &= \sum_i d\left((-1)^{i-1}F_i\right) \wedge dx_1 \wedge \dots \widehat{dx_i} \dots \wedge dx_n \\
&= \sum_i \left[\sum_j \frac{\partial}{\partial x_j}(-1)^{i-1}F_i dx_j\right] \wedge dx_1 \wedge \dots \widehat{dx_i} \dots \wedge dx_n \\
&= \sum_i (-1)^{i-1}\frac{\partial F_i}{\partial x_i} dx_i \wedge dx_1 \wedge \dots \widehat{dx_i} \dots \wedge dx_n \\
&= \sum_i \frac{\partial F_i}{\partial x_i} dx_1 \wedge \dots \wedge dx_n \\
&= \operatorname{div}(F) \cdot dx_1 \wedge \dots \wedge dx_n.
\end{aligned}
$$

(c)

$$\int_{\partial X} \eta = \int_X d\eta = \int_X d(\iota_F\omega) = \int_X \operatorname{div}(F) \cdot \omega$$

74. A differential form which is closed but not exact.

Consider on the punctured plane $\mathbb{R}^2 \setminus \{0\}$ the 1-form

$$\omega := -\frac{y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y.$$

(a) Show that ω is closed.

(b) Compute $\int_{\mathbb{S}^1} \omega$.

(c) Why does it follow from that ω is not exact?

Remark. Due to $\mathrm{d}(\mathrm{d}\eta) = 0$ we see that every exact form is closed. *Poincaré's Lemma* says that on *star-shaped* regions in $\mathbb{R}^n$ that the converse is also true: every closed form is exact. The example in this exercise shows that such a converse result cannot hold for general regions.

Solution.

(a)

$$d\omega = -\left(\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right) dy \wedge dx + \left(\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\right) dx \wedge dy = 0$$

(b) We use the parametrisation f and result from Exercise 43 on the last tutorial sheet:

$$\int_{\mathbb{S}^1} \omega = \int_0^{2\pi} -\frac{\sin t}{1}\, d(\cos t) + \frac{\cos t}{1}\, d(\sin t) = \int_0^{2\pi} \sin^2 t\, dt + \cos^2 t\, dt = 2\pi.$$

(c) By Stokes' theorem if ω were exact then this integral would be zero.

Additional Exercises

75. An integration.

Let $\omega = y\,\mathrm{d}x + z\,\mathrm{d}y$ be a 1-form on $\mathbb{R}^3$. Consider the restriction of ω to the 2-sphere $\mathbb{S}^2$, with the parametrisation

$$S^2 = \{\, (\sin(\varphi)\sin(\vartheta), \cos(\varphi)\sin(\vartheta), \cos(\vartheta)) \in \mathbb{R}^3 \,|\, \varphi \in [0, 2\pi), \vartheta \in [0, \pi]\,\}.$$

Verify through direct computation that Stokes' theorem holds for this case:

$$\int_{S^2} \mathrm{d}\omega = 0.$$

Solution. First,

$$d\omega = dy \wedge dx + dz \wedge dy.$$

We will also need to calculate the pullback by the parametrisation $f(\varphi, \vartheta) = (\sin(\varphi)\sin(\vartheta), \cos(\varphi)\sin($

$$\begin{aligned}
f^*dx &= d(\sin(\varphi)\sin(\vartheta)) = \cos(\varphi)\sin(\vartheta)d\varphi + \sin(\varphi)\cos(\vartheta)d\vartheta \\
f^*dy &= -\sin(\varphi)\sin(\vartheta)d\varphi + \cos(\varphi)\cos(\vartheta)d\vartheta \\
f^*dz &= -\sin(\vartheta)d\vartheta \\
f^*(dy \wedge dx) &= -\sin^2(\varphi)\sin(\vartheta)\cos(\vartheta)d\varphi \wedge d\vartheta + \cos^2(\varphi)\sin(\vartheta)\cos\vartheta d\vartheta \wedge d\varphi \\
&= -\sin(\vartheta)\cos(\vartheta)d\varphi \wedge d\vartheta \\
f^*(dz \wedge dy) &= -\cos(\varphi)\sin^2(\vartheta)d\vartheta \wedge d\varphi \\
&= \cos(\varphi)\sin^2(\vartheta)d\varphi \wedge d\vartheta
\end{aligned}$$

We can ignore sets of measure zero when pulling back using the parametrisation.

$$\begin{aligned}
\int_{S^2} \mathrm{d}\omega &= \int_{[0,2\pi]\times[0,\pi]} f^*\mathrm{d}\omega \\
&= \int_{[0,2\pi]\times[0,\pi]} \Big[-\sin(\vartheta)\cos(\vartheta) + \cos(\varphi)\sin^2(\vartheta) \Big] d\varphi \wedge d\vartheta \\
&= \int_0^{\pi} \left[\int_0^{2\pi} -\sin(\vartheta)\cos(\vartheta) + \cos(\varphi)\sin^2(\vartheta)\; d\varphi \right] d\vartheta \\
&= \int_0^{\pi} -2\pi\sin(\vartheta)\cos(\vartheta)\; d\vartheta = \int_0^{\pi} -\pi\sin(2\vartheta)\; d\vartheta = 0.
\end{aligned}$$

76. Volume forms on compact connected manifolds.

Let X be a compact connected orientable n-dimensional manifold without boundary, and suppose that ω is a non-vanishing n-form. Show that ω is not exact.

Hint. Calculate $\int_X \omega$ in two ways: with Stokes' theorem and with Definition 3.21.

Solution. Suppose that ω was exact: $\omega = d\eta$. Then by Stokes' theorem

$$0 = \int_{\emptyset} \eta = \int_{\partial X} \eta = \int_X d\eta = \int_X \omega.$$

On the other hand, from the definition of integration on manifolds, let $\{(U_k, \phi_k)\}$ be an oriented atlas of X and f_k the corresponding partition of unity. Without loss of generality, assume all the sets U_k are connected. Write $\omega = g_k d\phi_{k,1} \wedge \cdots \wedge d\phi_{k,n}$. Because g_k is non-vanishing, it has a definite sign on U_k. Because we are using an orientable atlas, all of the functions g_k have the same sign. Assume this sign is positive. Then

$$\int_X \omega = \sum_k \int_{\phi_k[U_k]} f_k(\phi_k^{-1}(x)) g_k(\phi_k^{-1}(x))\, dx_1 \ldots dx_n \geq \int_{\phi_0[U_0]} f_0(\phi_0^{-1}(x)) g_0(\phi_0^{-1}(x))\, dx_1 \ldots dx_n > 0$$

since the integral of a non-negative continuous function that is positive at a point must be positive. This is contradiction. Hence ω is not exact.