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Analysis III
12. Exercise: Integration

28 November 2022

Preparation Exercises

70. The pullback of differential forms.

(a) Let X, Y be manifolds of dimension n and $f : X \to Y$ a smooth map. Further take the standard local set-up of charts $\phi = (\phi_1, \ldots, \phi_n) : U \to \mathbb{R}^n$ and $\psi = (\psi_1, \ldots, \psi_n) : V \to \mathbb{R}^n$ on open sets $U \subset X$ and $V \subset Y$ with $f(U) \subset V$.

Show the following local formula for the pullback holds for every smooth function $g \in \mathcal{C}^\infty(V, \mathbb{R})$:

$$f^*(g\,\mathrm{d}\psi_1 \wedge \cdots \wedge \mathrm{d}\psi_n) = (g \circ f) \cdot \det\left(\frac{\partial(\psi_j \circ f \circ \phi^{-1})}{\partial x_i}\right) \cdot \mathrm{d}\phi_1 \wedge \cdots \wedge \mathrm{d}\phi_n\ .$$

Hint. Make use of the determinant formula for the evaluation of forms $\langle A_1 \wedge \cdots \wedge A_p\,,\, v_1 \otimes \ldots \otimes v_p\rangle = \det\big(A_i(v_j)\big)_{i,j}$, from page 71 of the script.

(b) Consider the *canonical volume form* on $\mathbb{R}^3$, namely $\omega := \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ and *spherical coordinates*

$$f : \mathbb{R}_+ \times [0, 2\pi) \times [0, \pi] \to \mathbb{R}^3,\ (r, \vartheta, \varphi) \mapsto \big(r\,\cos(\vartheta)\,\cos(\varphi)\,,\, r\,\cos(\vartheta)\,\sin(\varphi)\,,\, r\,\sin(\vartheta)\,\big).$$

Compute "ω in spherical coordinates", by which we mean the pullback $f^*\omega$.

71. Integration on the unit circle.

Let ω be a 1-form on the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$ and

$$f : \mathbb{R} \to \mathbb{S}^1,\ t \mapsto (\cos t, \sin t)$$

a paramterisation.

(a) Use the exercise "Null sets of manifolds" and Corollary 3.22 to calculate

$$\int_{\mathbb{S}^1} y\, dx$$

(b) Prove Stokes' theorem for $\mathbb{S}^1$. Actually, show the stronger result that ω is exact if and only if

$$\int_{\mathbb{S}^1} \omega = 0.$$

($\mathbb{S}^1$ is a manifold whose boundary is empty, so the right side of Stokes' theorem is zero.)

In Class Exercises

72. Null sets of manifolds.

Let M be an oriented manifold and Z a closed subset. Hence $M \setminus Z$ is also a manifold. We call Z a null set if for every coordinate chart $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ the set $\phi_\alpha[Z \cap U_\alpha]$ is a null set of $\mathbb{R}^n$. Prove that

$$\int_A \omega = \int_{A \setminus Z} \omega$$

73. The divergence theorem (aka Gauss' theorem).

Let $X \subset \mathbb{R}^n$ be a compact subset of $\mathbb{R}^n$ with $\overline{X^0} = X$ that is an n-dimensional manifold with boundary. It is know that X must be orientable and that $\omega := \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ is a volume form on X. Further, let a smooth $(n-1)$-form η on X be given.

(a) Show that there is a unique vector field $F \in \mathrm{Vec}^\infty(X)$ with $\eta = i_F\omega$.

(b) Write $F = (F_1, \ldots, F_n)$ for functions $F_1, \ldots, F_n \in C^\infty(X, \mathbb{R})$. Define the divergence operator $\mathrm{div}(F) \in C^\infty(X, \mathbb{R})$ as

$$\mathrm{div}(F) := \sum_{k=1}^{n} \frac{\partial F_k}{\partial x_k}.$$

Prove the following connection between the divergence operator and the exterior derivative:

$$\mathrm{d}(i_F\omega) = \mathrm{div}(F) \cdot \omega.$$

(c) Prove the divergence theorem:

$$\int_{\partial X} \eta = \int_X \mathrm{div}(F) \cdot \omega.$$

74. A differential form which is closed but not exact.

Consider on the punctured plane $\mathbb{R}^2 \setminus \{0\}$ the 1-form

$$\omega := -\frac{y}{x^2 + y^2}\,\mathrm{d}x + \frac{x}{x^2 + y^2}\,\mathrm{d}y.$$

(a) Show that ω is closed.

(b) Compute $\int_{\mathbb{S}^1} \omega$.

(c) Why does it follow from that ω is not exact?

Remark. Due to $\mathrm{d}(\mathrm{d}\eta) = 0$ we see that every exact form is closed. *Poincaré's Lemma* says that on *star-shaped* regions in $\mathbb{R}^n$ that the converse is also true: every closed form is exact. The example in this exercise shows that such a converse result cannot hold for general regions.

Additional Exercises

75. An integration.

Let $\omega = y\,\mathrm{d}x + z\,\mathrm{d}y$ be a 1-form on $\mathbb{R}^3$. Consider the restriction of ω to the 2-sphere $\mathbb{S}^2$, with the parametrisation

$$S^2 = \{\, (\sin(\varphi)\sin(\vartheta), \cos(\varphi)\sin(\vartheta), \cos(\vartheta)) \in \mathbb{R}^3 \mid \varphi \in [0, 2\pi), \vartheta \in [0, \pi] \,\}.$$

Verify through direct computation that Stokes' theorem holds for this case:

$$\int_{S^2} \mathrm{d}\omega = 0.$$

76. Volume forms on compact connected manifolds.

Let X be a compact connected orientable n-dimensional manifold without boundary, and suppose that ω is a non-vanishing n-form. Show that ω is not exact.

Hint. Calculate $\int_X \omega$ in two ways: with Stokes' theorem and with Definition 3.21.