

### Preparation Exercises

#### 66. The wedge product

In this exercise we will do some calculations with the antisymmetric algebras of  $V = \mathbb{R}^3$ . Let  $\{e_i\}$  be a basis of  $V$  and  $\{\alpha_i\}$  the corresponding dual basis of  $V'$ .

- (a) Show that  $\beta = \alpha_1 \otimes \alpha_2 - \alpha_2 \otimes \alpha_1$  belongs to  $\bigwedge^2 V'$ .
- (b) Consider the antisymmetrising operation  $\mathcal{A}$  defined in the proof of Theorem 3.4. Compute  $\mathcal{A}^1(\alpha_3) \in \bigwedge^1 V'$  and  $\mathcal{A}^2(\alpha_1 \otimes \alpha_2) \in \bigwedge^2 V'$ .
- (c) Compute  $\langle \beta, e_1 \otimes e_3 \rangle = \beta(e_1, e_3)$  using the definition of tensors as linear maps and also using the formula on page 71 of the script (notice that  $\beta = \alpha_1 \wedge \alpha_2$ ).
- (d) Compute the wedge product  $\beta \wedge \alpha_3$ .
- (e) In the proof of Theorem 3.4, it is proved that the wedge products span the space of antisymmetric tensors by a dimension count argument. Given an antisymmetric tensor, can you find an algorithm to write it as a sum of wedge products of basis elements?

#### Solution.

- (a) The only non-trivial element of the symmetric group  $S_2$  is the transposition (12). More generally, the symmetric group is generated by the  $n-1$  adjacent transpositions and it is sufficient to check the relation holds for these. The sign of transpositions is by definition  $-1$ .

$$(12).\beta = \alpha_2 \otimes \alpha_1 - \alpha_1 \otimes \alpha_2 = -\beta$$

- (b) The only element of  $S_1$  is the identity:

$$\mathcal{A}^1(\alpha_3) = \sum_{\sigma \in S_1} \text{sign}(\sigma)\sigma.\alpha_3 = 1 \text{ id}.\alpha_3 = \alpha_3.$$

The two elements of  $S_2$  are the identity and (12)

$$\begin{aligned} \mathcal{A}^2(\alpha_1 \otimes \alpha_2) &= \sum_{\sigma \in S_2} \text{sign}(\sigma)\sigma.(\alpha_1 \otimes \alpha_2) \\ &= \text{id}.\alpha_1 \otimes \alpha_2 - (12).\alpha_1 \otimes \alpha_2 \\ &= \alpha_1 \otimes \alpha_2 - \alpha_2 \otimes \alpha_1 = \beta. \end{aligned}$$

(c) Using the definition as linear maps

$$\beta(e_1, e_3) = \alpha_1(e_1) \cdot \alpha_2(e_3) - \alpha_2(e_1) \cdot \alpha_1(e_3) = 1 \cdot 0 - 0 \cdot 0 = 0.$$

On the other hand, we have that  $\beta = \frac{1}{1!1!} \mathcal{A}(\alpha_1 \otimes \alpha_2) = \alpha_1 \wedge \alpha_2$  Using the determinant formula

$$\langle \beta, e_1 \otimes e_3 \rangle = \begin{vmatrix} \alpha_1(e_1) & \alpha_1(e_3) \\ \alpha_2(e_1) & \alpha_2(e_3) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0.$$

The number is not so interesting, but notice if we expanded the determinant as a formula we would get the linear maps version.

(d)

$$\begin{aligned} \beta \wedge \alpha_3 &= \frac{1}{2!1!} \mathcal{A}^3(\beta \otimes \alpha_3) \\ &= \frac{1}{2} \mathcal{A}^3(\alpha_1 \otimes \alpha_2 \otimes \alpha_3 - \alpha_2 \otimes \alpha_1 \otimes \alpha_3) \\ \mathcal{A}^3(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) &= (1 - (12) - (13) - (23) + (123) + (132)) \cdot (\alpha_1 \otimes \alpha_2 \otimes \alpha_3) \\ &= \alpha_1 \otimes \alpha_2 \otimes \alpha_3 - \alpha_2 \otimes \alpha_1 \otimes \alpha_3 - \alpha_3 \otimes \alpha_2 \otimes \alpha_1 - \alpha_1 \otimes \alpha_3 \otimes \alpha_2 \\ &\quad + \alpha_2 \otimes \alpha_3 \otimes \alpha_1 + \alpha_3 \otimes \alpha_1 \otimes \alpha_2 \\ \mathcal{A}^3(\alpha_2 \otimes \alpha_1 \otimes \alpha_3) &= (1 - (12) - (13) - (23) + (123) + (132)) \cdot (\alpha_2 \otimes \alpha_1 \otimes \alpha_3) \\ &= \alpha_2 \otimes \alpha_1 \otimes \alpha_3 - \alpha_1 \otimes \alpha_2 \otimes \alpha_3 - \alpha_3 \otimes \alpha_1 \otimes \alpha_2 - \alpha_2 \otimes \alpha_3 \otimes \alpha_1 \\ &\quad + \alpha_1 \otimes \alpha_3 \otimes \alpha_2 + \alpha_3 \otimes \alpha_2 \otimes \alpha_1 \\ &= -\mathcal{A}^3(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) \\ \beta \wedge \alpha_3 &= \alpha_1 \otimes \alpha_2 \otimes \alpha_3 - \alpha_2 \otimes \alpha_1 \otimes \alpha_3 - \alpha_3 \otimes \alpha_2 \otimes \alpha_1 - \alpha_1 \otimes \alpha_3 \otimes \alpha_2 \\ &\quad + \alpha_2 \otimes \alpha_3 \otimes \alpha_1 + \alpha_3 \otimes \alpha_1 \otimes \alpha_2 \end{aligned}$$

(e) First we give an overview of the idea. Suppose we are given an antisymmetric tensor as a sum of pure tensors of basis elements. Our procedure is to choose one pure tensor from this, apply  $\mathcal{A}$  to it and then subtract the result. This must reduce the number of pure tensors in the sum, so after a finite number of iterations we have written our antisymmetric tensor as a sum of  $\mathcal{A}$  applied to pure tensors. But this is exactly a sum of wedge products.

More precisely, suppose that  $\beta \in \bigwedge^p V'$ . We can write it as

$$\beta = \sum \beta_{i_1, \dots, i_p} \alpha_{i_1} \otimes \dots \otimes \alpha_{i_p}.$$

Choose a non-zero term of this sum  $\beta_{j_1, \dots, j_p} \alpha_{j_1} \otimes \dots \otimes \alpha_{j_p}$  and apply  $\mathcal{A}^p$  to it to get the antisymmetric tensor  $A_1 := \beta_{j_1, \dots, j_p} \alpha_{j_1} \wedge \dots \wedge \alpha_{j_p}$ . Consider  $\beta - A_1$ . I claim that all terms whose indices are a permutation of  $\{j_1, \dots, j_p\}$  are zero. Suppose

for contradiction that its  $\sigma.\alpha_{j_1} \otimes \cdots \otimes \alpha_{j_p}$  term was non-zero. But  $\sigma.(\beta - A_1) = \text{sign } \sigma(\beta - A_1)$  because they are antisymmetric tensors. This means the coefficient of the  $\sigma.\alpha_{j_1} \otimes \cdots \otimes \alpha_{j_p}$  term is equal to  $\text{sign } \sigma$  times the coefficient of the  $\alpha_{j_1} \otimes \cdots \otimes \alpha_{j_p}$  term, which is  $\beta_{j_1, \dots, j_p} - \beta_{j_1, \dots, j_p} = 0$  by construction.

Thus  $\beta - A_1$  is again an antisymmetric tensor, with fewer non-zero terms. Iterating this procedure we get that

$$\beta = A_1 + \cdots + A_k = \sum \beta_{j_1, k, \dots, j_p, k} \alpha_{j_1, k} \wedge \cdots \wedge \alpha_{j_p, k}$$

is a sum of wedge products of basis elements.

Let us give an example of this procedure. Consider the antisymmetric tensor

$$\beta = 3\alpha_1 \otimes \alpha_2 - 2\alpha_1 \otimes \alpha_3 - 3\alpha_2 \otimes \alpha_1 + 2\alpha_2 \otimes \alpha_3 + 2\alpha_3 \otimes \alpha_1 - 2\alpha_3 \otimes \alpha_2.$$

Then we choose the  $\alpha_1 \otimes \alpha_2$  term

$$A_1 = 3\alpha_1 \wedge \alpha_2 = \mathcal{A}(3\alpha_1 \otimes \alpha_2) = 3\alpha_1 \otimes \alpha_2 - 3\alpha_2 \otimes \alpha_1$$

and so

$$\beta - A_1 = -2\alpha_1 \otimes \alpha_3 + 2\alpha_2 \otimes \alpha_3 + 2\alpha_3 \otimes \alpha_1 - 2\alpha_3 \otimes \alpha_2$$

does indeed have fewer terms. Now we choose the  $\alpha_1 \otimes \alpha_3$  term

$$A_2 = -2\alpha_1 \wedge \alpha_3 = -2\alpha_1 \otimes \alpha_3 + 2\alpha_3 \otimes \alpha_1$$

so

$$\beta - A_1 - A_2 = 2\alpha_2 \otimes \alpha_3 - 2\alpha_3 \otimes \alpha_2.$$

We can repeat this one more time with  $A_3 = 2\alpha_2 \wedge \alpha_3$  to exhaust the non-zero terms.

This gives at the end

$$\beta = A_1 + A_2 + A_3 = 3\alpha_1 \wedge \alpha_2 - 2\alpha_1 \wedge \alpha_3 + 2\alpha_2 \wedge \alpha_3.$$

## In Class Exercises

### 67. Dual 1-forms to a vector field (using dot product).

Let  $F \in \text{Vec}^\infty(\mathbb{R}^3)$  be a smooth vector field on  $\mathbb{R}^3$  that is nowhere vanishing. Find a tensor field  $\alpha$  of  $T^*\mathbb{R}^3$  (a 1-differential form on  $\mathbb{R}^3$ ), so that the kernel of  $\alpha$  at every point is orthogonal to  $F$ . Orthogonal means using the dot product of  $\mathbb{R}^3$

**Solution.** Let  $e_i$  be the standard basis of  $\mathbb{R}^3$  which gives a non-vanishing vector field. Thus we can write  $F(x) = \sum F_i(x)e_i$ . This induces dual basis fields  $\alpha_j$  on  $T'\mathbb{R}^3$ , which act as  $\alpha_j(F) = F_j(x)$ . Any 1-form  $\alpha$  can be written as  $\alpha(x) = \sum a_j(x)\alpha_j$  for smooth functions  $a_j$ .

What we require in this question is to find a form, such that  $G \cdot F = 0$  at every point if and only if  $\alpha(G) \equiv 0$ . Observe

$$\alpha(G)(x) = \sum a_j(x)G_j(x) = a(x) \cdot G(x)$$

where we treat the coefficients of  $\alpha$  as the coefficients of a vector field. Clearly then we should take  $\alpha(x) = \sum F_j(x)\alpha_j$  as the 1-form.

One point to see here is that for a vector field  $F$  in Euclidean space  $\langle F, \cdot \rangle$  is a 1-form. This explains why we also use inner product notation for the pairing between forms and fields.

## 68. Local representations of tensor fields.

Let  $X$  be an  $n$ -dimensional smooth manifold and  $f$  a tensor field in  $T_p^q X$ . To simplify the notation, we will only consider the case  $p = 0, q = 2$ , but for other tensor spaces everything holds completely analogously. Further, let  $(U, \phi)$  be a chart of  $X$  and denote the components of  $\phi$  by  $\phi_1, \dots, \phi_n : U \rightarrow \mathbb{R}$ . These induce 1-forms  $\alpha_k := d\phi_k$  in  $T_0^1 U \subset T_0^1 X$ . By definitions these 1-forms act as  $\alpha_k(x)(v) = T_x(\phi_k)(v)$  for every  $x \in U$  and  $v \in T_x X$ .

General Hint. It will be useful to consider the (local) vector fields  $E_k$  on  $U$

$$E_k(x) = T_x(\phi)^{-1}(e_k).$$

These are often called the coordinate vector fields.

(a) Show there exist functions  $f_{k,l} : U \rightarrow \mathbb{R}$ , so that

$$f|U = \sum_{k,l=1}^n f_{k,l} \cdot \alpha_k \otimes \alpha_l.$$

More precisely, we mean that for all  $x \in U$  and  $v, w \in T_x X$

$$f(x)(v, w) = \sum_{k,l=1}^n f_{k,l}(x) \alpha_k(x)(v) \alpha_l(x)(w).$$

Show further that these functions are unique. This way of describing  $f$  is called representing  $f$  in local coordinates. The functions  $f_{k,l}$  are called the coefficient functions.

- (b) Show that  $f|U$  is smooth exactly when the functions  $f_{k,l}$  are all smooth.  
(c) Show that  $f$  is a 2-form exactly when the coefficients are antisymmetric:  $f_{k,l} = -f_{l,k}$ .

**Solution.**

- (a) Note that the 1-forms form a local trivialisation of  $T_0^1U$ : Due to the relation

$$d\phi_i(x)(E_j(x)) = T_x(\phi_i)(T_x(\phi)^{-1}(e_j)) = T_x(\pi_i \circ \phi)(T_x(\phi)^{-1}(e_j)) = \pi_i(e_j) = \delta_{i,j}$$

it follows that they are non-vanishing at every point and linearly independent. Therefore  $\alpha_i(x) \otimes \alpha_j(x)$  form a basis for  $T_{0,x}^2X$  and hence every  $(0, 2)$  tensor  $f(x)$  is a unique linear combination of them.

- (b) If the coefficient functions are all smooth, so is the sum, because the basis sections  $\alpha_j$  are smooth since they are the exterior derivative of smooth functions (Theorem 3.14). In the converse direction, the coefficient functions may be realised as the projections of  $f$  with respect to the coordinate chart, but this is the composition of smooth functions.

- (c)  $f$  is a 2-form when it is antisymmetric in every fibre. So the question is really: when is a 2-tensor  $\sum a_{i,j}e_i \otimes e_j$  antisymmetric? There is only one non-trivial element of  $S_2$  to consider:

$$-\sum a_{i,j}e_i \otimes e_j = (12) \cdot (\sum a_{i,j}e_i \otimes e_j) = \sum a_{i,j}e_j \otimes e_i = \sum a_{j,i}e_i \otimes e_j$$

from which we require  $a_{i,j} = -a_{j,i}$ .

## 69. Closed and exact differential forms.

A  $p$ -form  $\omega$  on a manifold  $X$  is called closed if  $d\omega = 0$ , and it is called exact if there is a  $(p-1)$ -form  $\theta$  on  $X$  with  $\omega = d\theta$ .

Consider  $X = \mathbb{R}^3$  and let  $x, y, z : \mathbb{R}^3 \rightarrow \mathbb{R}$  be the usual coordinate functions. Investigate whether the following forms are closed and or exact.

- (a)  $\omega = yz \, dx + xz \, dy + xy \, dz$   
(b)  $\omega = x \, dx + x^2y^2 \, dy + yz \, dz$   
(c)  $\omega = 2xy^2 \, dx \wedge dy + z \, dy \wedge dz$

**Solution.** Before we begin, recall Theorem 3.14(iii):  $d(df) = 0$  for a function  $f$ , and for part (c) the more general result of Theorem 3.15(i):  $d(d\alpha) = 0$  for any differential form  $\alpha$ . This implies that any exact form is closed, or that a form cannot be exact if it is not closed. This can be an effective test.

(a)

$$\begin{aligned}d(yz dx + xz dy + xy dz) &= d(yz) \wedge dx + d(xz) \wedge dy + d(xy) \wedge dz \\&= (ydz + zdy) \wedge dx + (xdz + zdx) \wedge dy + (xdy + ydx) \wedge dz \\&= ydz \wedge dx + zdy \wedge dx + xdz \wedge dy + zdx \wedge dy + xdy \wedge dz + ydx \wedge dz \\&= -ydx \wedge dz - zdx \wedge dy - xdy \wedge dz + zdx \wedge dy + xdy \wedge dz + ydx \wedge dz \\&= 0.\end{aligned}$$

So  $\omega$  is closed. That means it might be exact (but in general it doesn't have to be). In this case it's relatively easy to see that  $\omega = d(xyz)$ .

(b)

$$\begin{aligned}d(x dx + x^2 y^2 dy + yz dz) &= dx \wedge dx + d(x^2 y^2) \wedge dy + d(yz) \wedge dz \\&= 0 + (2xy^2 dx + 2x^2 y dy) \wedge dy + (z dy + y dz) \wedge dz \\&= 2xy^2 dx \wedge dy + 0 + z dy \wedge dz + 0.\end{aligned}$$

So  $\omega$  is not closed, and therefore cannot be exact.

(c)

$$\begin{aligned}d(2xy^2 dx \wedge dy + z dy \wedge dz) &= d(2xy^2) \wedge dx \wedge dy + dz \wedge dy \wedge dz \\&= (2y^2 dx + 4xy dy) \wedge dx \wedge dy + 0 \\&= 0.\end{aligned}$$

So  $\omega$  is closed. Let  $\theta = adx + bdy + cdz$  for functions  $a, b, c$ . Then we must try to solve  $d\theta = \omega$ . I will use subscripts for the partial derivatives.

$$\begin{aligned}d\theta &= (a_x dx + a_y dy + a_z dz) \wedge dx + (b_x dx + b_y dy + b_z dz) \wedge dy + (c_x dx + c_y dy + c_z dz) \wedge dz \\&= -a_y dx \wedge dy - a_z dx \wedge dz + b_x dx \wedge dy - b_z dy \wedge dz + c_x dx \wedge dz + c_y dy \wedge dz \\&= (b_x - a_y) dx \wedge dy + (c_x - a_z) dx \wedge dz + (c_y - b_z) dy \wedge dz.\end{aligned}$$

Hence we must find functions that satisfy  $b_x - a_y = 2xy^2$ ,  $c_x - a_z = 0$  and  $c_y - b_z = z$ . Assume that  $c = 0$ . That means that  $a$  is a function of  $x, y$  only and  $b(x, y, z) = -\frac{1}{2}z^2 + \tilde{b}(x, y)$ . Then we only need to further satisfy the first equation

$$\tilde{b}_x - a_y = 2xy^2.$$

This can be done by choosing  $b = -\frac{1}{2}z^2 + x^2 y^2$  and  $a = 0$ . So  $\theta = (-\frac{1}{2}z^2 + x^2 y^2) dy$  shows that  $\omega$  is exact.

Note, this is not the only solution: if  $f$  is any function then  $d(\theta + df) = d\theta = \omega$ .