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Analysis III
10. Exercise: Multilinear Maps and Tensors

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In the exercises below, let $V, V_1, \ldots, V_n, W$ be finite dimensional normed vector spaces over $\mathbb{K}$. We will ignore the norms though, because in finite dimensions all norms are equivalent and linear maps are automatically continuous.

We will make use of the Kronecker notation

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Preparation Exercises

58. The difference between linear and multilinear.

Give an example of a multilinear map in $\mathcal{L}(\mathbb{R}, \mathbb{R}; \mathbb{R})$. Does it belong to $\mathcal{L}(\mathbb{R}^2; \mathbb{R})$?

Solution. Instead of give an example, we find all possible examples. Let $A \in \mathcal{L}(\mathbb{R}, \mathbb{R}; \mathbb{R})$. Let $a = A(1, 1)$. Then by multilinearity, $A(x, y) = xA(1, y) = xyA(1, 1) = xya$. Thus A is completely determined by its value at $(1, 1)$. Even though we can consider A as a function from $\mathbb{R}^2$ to $\mathbb{R}$, it is not a linear map in $\mathcal{L}(\mathbb{R}^2; \mathbb{R})$ because for any scalar λ

$$A(\lambda x, \lambda y) = \lambda A(x, \lambda y) = \lambda^2 A(x, y),$$

so instead it is quadratic with respect to the vector space $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

59. The dual space and matrices.

Recall that the dual of a vector space V is defined to be $V' := \mathcal{L}(V; \mathbb{K})$. Consider a basis $\{e_i\}$ of V. Show that the *dual basis* $\{A_i \in V'\}$ defined by $A_i(e_j) = \delta_{i,j}$ is indeed a basis for V'. (Note, to define a linear map, it is enough to give its values on a basis of the domain.)

Suppose further that $\{f_i\}$ is a basis of W. Define $B_{i,j} \in \mathcal{L}(V; W)$ by $B_{i,j}(e_k) = f_i \delta_{j,k}$. Argue that these elements form a basis of $\mathcal{L}(V; W)$. Explain this result in terms of matrices.

Solution. A set of vectors is a basis when its elements are linearly independent and spanning. Suppose that there are scalars c_i such that $\sum c_i A_i = 0 \in V'$. Evaluating this at e_j shows that $c_j = 0$. Hence $\{A_i\}$ is linearly independent. Next, choose any element $A \in V'$. Consider the difference

$$A - \sum_i A(e_i) A_i \in V'.$$

If we evaluate this difference at e_j, then we see the result is zero. Hence this difference is the zero function in V'. In other words $A = \sum_i A(e_i)A_i$. This shows that $\{A_i\}$ span V'.

For $\{B_{i,j}\}$ essentially the same ideas work, but the algebra is a little more difficult. Suppose we had a linear dependence $\sum_{i,j} c_{i,j}B_{i,j} = 0$. Evaluating this at e_k shows that

$$0 = \sum_{i,j} c_{i,j}B_{i,j}(e_k) = \sum_{i,j} c_{i,j}f_i\delta_{j,k} = \sum_i c_{i,k}f_i.$$

But the $\{f_i\}$ are a basis of W, and so it must be that $c_{i,k} = 0$ for all i. Since this holds for all k, all the scalars vanish. We have therefore shown that $\{B_{i,j}\}$ are linearly independent.

As to whether they are spanning, choose any $B \in \mathcal{L}(V;W)$ we know that there exist scalars $b_{i,j}$ such that $B(e_k) = \sum_i b_{i,k}f_i$. But then the difference

$$B(e_k) - \sum_{i,j} b_{i,j}B_{i,j}(e_k) = \sum_i b_{i,k}f_i - \sum_i b_{i,k}f_i = 0.$$

Since this holds for all k, we conclude that $B = \sum_{i,j} b_{i,j}B_{i,j}$.

The connection to matrices is that $B_{i,j}$ is essentially the matrix with a 1 in the (i,j) position and zeroes elsewhere. Such a matrix acts on the j^{th} basis vector of V to give the i^{th} basis vector of W, and acts on all other basis vectors to give zero. The proof we just gave shows that all linear maps between V and W can be represented as a matrix. If we view $\mathbb{K}$ as a one-dimensional vector space over itself with 1 as a basis vector, then we have shown that V' is represented by a matrix that has only one row. The dual element A_i has a 1 in the i^{th} position and the remaining elements are zero. For this reason, dual vectors are sometimes called row vectors.

In Class Exercises

60. An iterative definition of multilinear maps.

We know that the space of linear maps is itself a vector space. Explain why the space of multilinear maps $\mathcal{L}(V_1, V_2; W)$ is isomorphic to $\mathcal{L}(V_1; \mathcal{L}(V_2; W))$.

Solution. Let $\Phi : \mathcal{L}(V_1, V_2; W) \to \mathcal{L}(V_1; \mathcal{L}(V_2; W))$ be

$$\Phi(A)(v_1)(v_2) = A(v_1, v_2)$$

for $v_1 \in V_1$ and $v_2 \in V_2$. For every v_1 we have a linear map $B = \Phi(A)(v_1)$ in $\mathcal{L}(V_2; W)$ since

$$B(av_2 + bv_2') = A(v_1, av_2 + bv_2') = aA(v_1, v_2) + bA(v_1, v_2')$$

by multilinearity. Likewise is $\Phi(A)$ a linear map between V_1 and $\mathcal{L}(V_2; W)$ since

$$\begin{aligned}\Phi(A)(av_1 + bv_1')(v_2) &= A(av_1 + bv_1', v_2) = aA(v_1, v_2) + bA(v_1', v_2)\\ &= a\Phi(A)(v_1)(v_2) + b\Phi(A)(v_1')(v_2).\end{aligned}$$

These two properties shows that Φ is well defined. It is linear because evaluation is linear:

$$\begin{aligned}\Phi(aA + bA')(v_1)(v_2) &= (aA + bA')(v_1, v_2) = aA(v_1, v_2) + bA'(v_1, v_2)\\ &= a\Phi(A)(v_1)(v_2) + b\Phi(A')(v_1)(v_2).\end{aligned}$$

Finally, it's easy to see that the inverse is

$$\Phi^{-1}(C)(v_1, v_2) = C(v_1)(v_2).$$

Together this shows that Φ is an isomorphism.

This map Φ, believe it or not, has a special name. It is called *currying*, `https://en.wikipedia.org/wiki/Currying`, named after Haskell Curry. It is sometimes called *partial evaluation*, which is closely related.

61. An isomorphism between $\mathcal{L}(V; W)$ and $\mathcal{L}(V, W'; \mathbb{K})$.

Use the iterative definition of multilinear maps to give an isomorphism between $\mathcal{L}(V; W)$ and $\mathcal{L}(V, W'; \mathbb{K})$.

Solution. Recall the fact that the dual of the dual is canonically isomorphic to the original space (at least, this is always true in finite dimensions). Then

$$\mathcal{L}(V, W'; \mathbb{K}) \cong \mathcal{L}(V, \mathcal{L}(W'; \mathbb{K})) \cong \mathcal{L}(V, W'') \cong \mathcal{L}(V, W).$$

62. Dimension of $\mathcal{L}(V_1, \ldots, V_n; W)$.

Show that

$$\dim \mathcal{L}(V_1, \ldots, V_n; W) = \dim(V_1) \cdot \ldots \cdot \dim(V_n) \cdot \dim(W)$$

Solution. As a base case, $\mathcal{L}(V_1; W)$ is the familiar space of linear maps from V_1 to W, each of which can be written as a $\dim W$ rows by $\dim V_1$ columns matrix. Thus it is a vector space of $\dim V_1 \cdot \dim W$. Next, by the iterative definition of multilinear maps $\mathcal{L}(V_1, \ldots, V_n; W) \cong \mathcal{L}(V_1, \mathcal{L}(V_2, \ldots, V_n; W))$. The formula follows by induction.

63. Tensor spaces.

Prove the following isomorphisms:

(a) $\mathcal{L}(V;W) \cong V' \otimes W$.

(b) $V_1 \otimes V_2 \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3) \cong (V_1 \otimes V_2) \otimes V_3$

(c) $\mathcal{L}(V_1, \ldots, V_n; W) \cong \mathcal{L}(V_1 \otimes \ldots \otimes V_n; W)$

Solution.

(a) This is probably the most useful isomorphism, because it enables us to reduce spaces of linear maps to tensor products, and I find tensor products easier. Simply $V' \otimes W = \mathcal{L}(V, W'; \mathbb{K}) \cong \mathcal{L}(V; W)$.

(b) We will prove the first isomorphism. By definition $V_1 \otimes V_2 \otimes V_3 = \mathcal{L}(V_1', V_2', V_3'; \mathbb{K})$. On the other hand, we have seen that the space of multilinear maps can be understood inductively as linear maps into the space of multilinear maps. Hence

$$\begin{aligned}\mathcal{L}(V_1', V_2', V_3'; \mathbb{K}) \cong \mathcal{L}(V_1'; \mathcal{L}(V_2', V_3'; \mathbb{K})) = \mathcal{L}(V_1'; V_2 \otimes V_3) \cong \mathcal{L}(V_1', (V_2 \otimes V_3)'; \mathbb{K}) \\ = V_1 \otimes (V_2 \otimes V_3).\end{aligned}$$

(c) First, let us show that dualising distributes over the tensor product: $(V \otimes W)' = V' \otimes W'$. This follows since

$$(V \otimes W)' \cong \mathcal{L}(V'; W)' = \mathcal{L}(W; V') \cong \mathcal{L}(V, W; \mathbb{K}) = V' \otimes W'.$$

(Perhaps it is also a good exercise as to why the dual of the linear maps from V to W is the linear maps from W to V. Also called the transpose of a map.)

We can now prove the exercise. I'll show only the proof in the case $n = 2$, higher n follow similarly by induction.

$$\begin{aligned}\mathcal{L}(V_1, V_2; W) &\cong \mathcal{L}(V_1; \mathcal{L}(V_2; W)) = \mathcal{L}(V_1; V_2' \otimes W) = V_1' \otimes (V_2' \otimes W) \\ &\cong V_1' \otimes V_2' \otimes W \\ \mathcal{L}(V_1 \otimes V_2; W) &\cong \mathcal{L}(V_1 \otimes V_2, W'; \mathbb{K}) = (V_1 \otimes V_2)' \otimes W \cong (V_1' \otimes V_2') \otimes W \\ &\cong V_1' \otimes V_2' \otimes W\end{aligned}$$

64. The tensor product.

(a) Prove or disprove:

(i) the tensor product of vectors

$$V_1 \times \ldots \times V_n \to V_1 \otimes \ldots \otimes V_n, \; (v_1, \ldots, v_n) \mapsto v_1 \otimes \ldots \otimes v_n$$

is commutative in the case $V_1 = \ldots = V_n$.

(ii) every vector in $V_1 \otimes \ldots \otimes V_n$ is *pure (coherent).*

Solution.

(i) This is false, the tensor product is not commutative even when the vector spaces are all the same. Consider $n = 2$ and $V = \mathbb{R}^2$ with the standard basis e_1, e_2. Let the dual space V' have the dual basis A_1, A_2. By the construction of the double dual, V acts on V' by $v(A) := A(v)$. Let's apply Definition 3.2 to the following two tensors

$$\begin{aligned} &e_1 \otimes e_2, e_2 \otimes e_1 : V' \times V' \to \mathbb{R} \\ &e_1 \otimes e_2(A_1, A_2) := e_1(A_1) \cdot e_2(A_2) = A_1(e_1) \cdot A_2(e_2) = 1, \\ &e_2 \otimes e_1(A_1, A_2) := e_2(A_1) \cdot e_1(A_2) = A_1(e_2) \cdot A_2(e_1) = 0, \end{aligned}$$

so clearly they are different tensors.

(ii) This is false. Let's continue the example from the previous part. I claim that $t = e_1 \otimes e_2 - e_2 \otimes e_1$ is not a pure tensor. Let it act on two arbitrary vectors of V', namely $A = a_1 A_1 + a_2 A_2$ and $B = b_1 A_1 + b_2 A_2$:

$$t(A, B) = e_1(A) e_2(B) - e_2(A) e_1(B) = a_1 b_2 - a_2 b_1.$$

However, a pure tensor would produce

$$\begin{aligned} (c_1 e_1 + c_2 e_2) \otimes (d_1 e_1 + d_2 e_2)(A, B) &= (c_1 a_1 + c_2 a_2)(d_1 b_1 + d_2 b_2) \\ &= c_1 d_1 a_1 b_1 + c_2 d_1 a_2 b_1 + c_1 d_2 a_1 b_2 + c_2 d_2 a_2 b_2 \end{aligned}$$

So we would need for $c_1 d_2 = 1$ and $c_2 d_1 = -1$ but also $c_1 d_1 = 0$. This is not possible.

(b) Show that in $V_1 \otimes \ldots \otimes V_n$ the linear span of the pure tensors is $V_1 \otimes \ldots \otimes V_n$, ie. every element of $V_1 \otimes \ldots \otimes V_n$ is a finite linear combination of the pure tensors.

Solution. For each $j = 1, \ldots, n$ let $\{e_{i,j}\}_{i=1,\ldots,\dim V_j}$ be a basis of V_j and $\{A_{i,j}\}_{i=1,\ldots,\dim V_j'}$ be the dual basis of V_j'. By multilinearity, an element A of $\mathcal{L}(V_1, \ldots, V_n; \mathbb{K})$ is exactly determined by its values $A(e_{i_1,1}, e_{i_2,2}, \ldots, e_{i_n,n})$. But

$$A_{j_1,1} \otimes \cdots \otimes A_{j_n,n}(e_{i_1,1}, e_{i_2,2}, \ldots, e_{i_n,n}) = \delta_{i_1,j_1} \cdot \cdots \cdot \delta_{i_n,j_n}.$$

This allows us to write

$$A = \sum_{i_1=1}^{\dim V_1} \sum_{i_2=1}^{\dim V_2} \cdots \sum_{i_n=1}^{\dim V_n} A(e_{i_1,1}, e_{i_2,2}, \dots, e_{i_n,n}) A_{i_1,1} \otimes \cdots \otimes A_{i_n,n}.$$

Thus we have written every element of $\mathcal{L}(V_1, \dots, V_n; \mathbb{K}) = V_1' \otimes \dots V_n'$ as a sum of pure tensors.

Additional Exercises

65. Riemannian metric.

Let X be a manifold. Let $L(TX, TX; \mathbb{R})$ denote the vector bundle whose fibre over $x \in X$ is the $\mathbb{R}$-vector space of bilinear forms $T_xX \times T_xX \to \mathbb{R}$. A Riemannian metric (or simply a metric) on X is a global smooth section G of this vector bundle, such that $g(x)$ is a scalar product on T_xX for ever $x \in X$ (it is symmetric and positive definite).

Show that every manifold has a Riemannian metric.

Hint. Choose a cover of X by coordinate charts. Construct a Riemannian metric in each coordinate chart. 'Glue' them all together using a partition of unity.

Solution. Let us first do this in a single coordinate chart $\phi : U \to \mathbb{R}^n$. Then we know that $T(\phi)$ is a diffeomorphism between TU and TR^n. This has an obvious Riemannian metric, namely the dot product. Explicitly, if $v, w \in T_xU$, then we define

$$g(x)(v, w) = T_x(\phi)v \cdot T_x(\phi)w.$$

We see that this very much depends on the choice of chart.

Now let X be covered by an atlas $\mathcal{A}$ and let (α_α) be a subordinate partition of unity. In each coordinate neighbourhood U_α we have a Riemannian metric g_α. Let $g(x) = \sum \alpha_\alpha(x) g_\alpha(x)$. This is a well-defined global section, because α_α vanishes outside U_α and at any point at most finitely many of the terms are non-zero. Bilinearity and symmetry are also immediate, because the sum of symmetric bilinear forms is again a symmetric bilinear form. It remains to show positive definiteness. But the partition of unity is non-negative, so $g(x)$ must be non-negative. Suppose that $v \in T_xX$ is a non-zero vector. There must be at least one α_α that does not vanish at x because they sum to 1, so it follows that

$$g(x)(v, v) = \sum \alpha_\alpha(x) g_\alpha(x)(v, v) \geq \alpha_0(x) g_0(x)(v, v) > 0.$$

This shows positive definiteness.

Terminology

köherent = coherent. In English, we called these tensors pure, simple, or elementary.