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Analysis III

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10. Exercise: Multilinear Maps and Tensors

In the exercises below, let $V, V_1, \ldots, V_n, W$ be finite dimensional normed vector spaces over $\mathbb{K}$. We will ignore the norms though, because in finite dimensions all norms are equivalent and linear maps are automatically continuous.

We will make use of the Kronecker notation

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Preparation Exercises

58. The difference between linear and multilinear.

Give an example of a multilinear map in $\mathcal{L}(\mathbb{R}, \mathbb{R}; \mathbb{R})$. Does it belong to $\mathcal{L}(\mathbb{R}^2; \mathbb{R})$?

59. The dual space and matrices.

Recall that the dual of a vector space V is defined to be $V' := \mathcal{L}(V; \mathbb{K})$. Consider a basis $\{e_i\}$ of V. Show that the *dual basis* $\{A_i \in V'\}$ defined by $A_i(e_j) = \delta_{i,j}$ is indeed a basis for V'. (Note, to define a linear map, it is enough to give its values on a basis of the domain.)

Suppose further that $\{f_i\}$ is a basis of W. Define $B_{i,j} \in \mathcal{L}(V; W)$ by $B_{i,j}(e_k) = f_i \delta_{j,k}$. Argue that these elements form a basis of $\mathcal{L}(V; W)$. Explain this result in terms of matrices.

In Class Exercises

60. An iterative definition of multilinear maps.

We know that the space of linear maps is itself a vector space. Explain why the space of multilinear maps $\mathcal{L}(V_1, V_2; W)$ is isomorphic to $\mathcal{L}(V_1; \mathcal{L}(V_2; W))$.

61. An isomorphism between $\mathcal{L}(V; W)$ and $\mathcal{L}(V, W'; \mathbb{K})$.

Use the iterative definition of multilinear maps to give an isomorphism between $\mathcal{L}(V; W)$ and $\mathcal{L}(V, W'; \mathbb{K})$.

62. Dimension of $\mathcal{L}(V_1, \ldots, V_n; W)$.

Show that

$$\dim \mathcal{L}(V_1, \ldots, V_n; W) = \dim(V_1) \cdot \ldots \cdot \dim(V_n) \cdot \dim(W)$$

63. Tensor spaces.

Prove the following isomorphisms:

(a) $\mathcal{L}(V; W) \cong V' \otimes W$.

(b) $V_1 \otimes V_2 \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3) \cong (V_1 \otimes V_2) \otimes V_3$

(c) $\mathcal{L}(V_1, \ldots, V_n; W) \cong \mathcal{L}(V_1 \otimes \ldots \otimes V_n; W)$

64. The tensor product.

(a) Prove or disprove:

(i) the tensor product of vectors

$$V_1 \times \ldots \times V_n \to V_1 \otimes \ldots \otimes V_n,\ (v_1, \ldots, v_n) \mapsto v_1 \otimes \ldots \otimes v_n$$

is commutative in the case $V_1 = \ldots = V_n$.

(ii) every vector in $V_1 \otimes \ldots \otimes V_n$ is *pure (coherent)*.

(b) Show that in $V_1 \otimes \ldots \otimes V_n$ the linear span of the pure tensors is $V_1 \otimes \ldots \otimes V_n$, ie. every element of $V_1 \otimes \ldots \otimes V_n$ is a finite linear combination of the pure tensors.

Additional Exercises

65. Riemannian metric.

Let X be a manifold. Let $L(TX, TX; \mathbb{R})$ denote the vector bundle whose fibre over $x \in X$ is the $\mathbb{R}$-vector space of bilinear forms $T_xX \times T_xX \to \mathbb{R}$. A Riemannian metric (or simply a metric) on X is a global smooth section G of this vector bundle, such that $g(x)$ is a scalar product on T_xX for ever $x \in X$ (it is symmetric and positive definite).

Show that every manifold has a Riemannian metric.

Hint. Choose a cover of X by coordinate charts. Construct a Riemannian metric in each coordinate chart. 'Glue' them all together using a partition of unity.

Terminology

köherent = coherent. In English, we called these tensors pure, simple, or elementary.