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Analysis III
9. Exercise: Flows and Integral Curves

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Preparation Exercises

52. Examples of integral curves and flows.

Let F be a smooth vector field on $\mathbb{R}^2$ given by

$$F(x, y) = (-y, x)$$

(a) Find the maximal integral curves of F.

(b) Write down the maximal flow of F.

(c) Consider the restriction of F to $\mathbb{S}^1$. What are the integral curves and maximal flow?

Solution.

(a) If we draw F, we see that it is a circular vector field. We can immediately write down the solution to the flow equation:

$$\gamma'(t) = F(\gamma(t)), \text{ with } \gamma(0) = (x_0, y_0),$$

namely $\gamma(t) = (r\cos(t-t_0), r\sin(t-t_0))$ with $r = \sqrt{x_0^2 + y_0^2}$ and $(x_0, y_0) = \gamma(0)$. We could also obtain this by solving the differential equation. This is maximal because the domain of the curve is $\mathbb{R}$.

(b) The flow is $\phi(t, (x_0, y_0)) := (r(x_0, y_0)\cos(t - t_0(x_0, y_0)), r(x_0, y_0)\sin(t - t_0(x_0, y_0)))$, just the solution of the differential equation with the dependence of the solution on the initial condition as part of the function.

But this is ugly. The flow looks nicer when you write it using rotation matrices, because this separates the initial conditions in a clear way:

$$\gamma(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

So then we can write

$$\phi(t, (x_0, y_0)) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Either way, this flow is defined for all $t \in \mathbb{R}$ so it is maximal.

(c) Integral curves and flows are well behaved under restrict if the vector field restricts to the submanifold. So the integral curves of $\mathbb{S}^1$ are just those curves γ which begin on a point of $\mathbb{S}^1$ (and notice that they then stay in $\mathbb{S}^1$). The flow is just the restriction of the flow of $\mathbb{R}^2$.

53. A Flow on $\mathbb{S}^2$.

Consider the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. Define a vector field $\mathbb{F} : \mathbb{S}^2 \to \mathbb{R}^3$ by

$$F(x, y, z) = (-y, x, 0).$$

(a) Show that F is a vector field on S^2 (using the identification that comes from the inclusion map $\iota : \mathbb{S}^2 \to \mathbb{R}^3$).

(b) What are the integral curves of F?

(c) Determine the maximal flow ψ of F.

(d) Let $M := \mathbb{S}^2 \setminus \{(1, 0, 0)\}$. Find an open neighbourhood W of $\{0\} \times M$ in $\mathbb{R} \times M$ so that $\psi|W$ is a flow on M. Is $\psi|W$ a global flow on M?

Solution.

(a) Because $F \cdot (x, y, z) = yx - xy = 0$, we understand that it corresponds to a vector field on $\mathbb{S}^2$ using the map $T(\iota)$.

(b) Firstly, we can work on all of $\mathbb{R}^3$ and then because F is a vector field of $\mathbb{S}^2$ the restriction will give the solution on $\mathbb{S}^2$. Because the z-component of F is zero, the z-component of a point must be constant during its flow. Thus the question reduces to the previous exercise. We have the curves

$$\gamma(t) = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}.$$

(c) From the previous part

$$\phi(t, (x, y, z)) = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Notice that if $p \in \mathbb{S}^2$ then $\psi(t, p) \in \mathbb{S}^2$ for all time, so it does give a flow on $\mathbb{S}^2$ as we claimed it would. It is defined for all times, so maximal.

(d) The issue with removing a point of $\mathbb{S}^2$ is that the flow may want to move to this point. Therefore let us understand which points p flow into $(1, 0, 0)$, ie $\psi(t, p) = (1, 0, 0)$ for some t. Due to the time-homomorphism property of flows (Definition 2.8(ii)), these points are just

$$p = \phi(-t, (1, 0, 0)) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}.$$

Thus, all points of the equator want to flow through $(1, 0, 0)$ at some time. This means that there cannot be a global flow of this field on M, no matter how we choose W.

To find W we need to know an amount of time a point can flow before it tries to go to through the removed point $(1, 0, 0)$, not necessarily the maximum amount of time. There are many ways to do this. Observe that the difference in longitude between any point $(x, y, 0)$ on the equator and $(1, 0, 0)$ is greater than $1 - x$. Thus for any point on the equator $\psi(t, (x, y, z)) \neq (1, 0, 0)$ for all $t \in (-(1 - x), (1 - x))$. In fact, notice that this is true for all points of M and $1 - x$ is a positive function on M. Thus define

$$W = \Big\{(t, (x, y, z)) \in \mathbb{R} \times M \,\Big|\, t \in (-(1 - x), (1 - x))\Big\},$$

and $\psi|_W$ is a local flow of M.

In Class Exercises

54. An example of an non-complete vector field.

Let

$$W := \{\, (t, (x, y)) \in \mathbb{R} \times \mathbb{R}^2 \,|\, 2\,(x^2 + y^2) \cdot t < 1 \,\}$$

and

$$\psi : W \to \mathbb{R}^2,\ (t, (x, y)) \mapsto \frac{1}{\sqrt{1 - 2\,(x^2 + y^2) \cdot t}} \cdot (x, y).$$

(a) Show that ψ is a flow on $\mathbb{R}^2$.

(b) Determine the corresponding vector field $F \in \mathrm{Vec}^\infty(\mathbb{R}^2)$.

(c) Explain why ψ is the maximal flow of F, and why F is not a complete vector field.

Solution.

(a) Let us check the properties in Definition 2.8. For each point of $\mathbb{R}^2$ we see that $2(x^2 + y^2) \cdot t < 1$ is an open interval containing zero. Notice that we can also write the set W as those points (t, p) with $2\|p\|^2\, t < 1$. So if $(s, p) \in W$ and $(t, \phi(s, p)) \in W$ then

$$1 > 2\|\psi(s, p)\|^2\, t = 2 \cdot \frac{1}{1 - 2\|p\|^2 \cdot s}\|p\|^2 \cdot t$$

$$1 - 2\|p\|^2\, s > 2\|p\|^2\, t$$

$$1 > 2\|p\|^2(t + s),$$

which shows $(t+s,p)$ belongs to W also. Thus $\psi(t,\psi(s,p))$ is well-defined and we can compute

$$\begin{aligned}\psi(t,\psi(s,p)) &= \left(1-2\|\psi(s,p)\|^2\, t\right)^{-0.5}\psi(s,p)\\ &= \left(1-\frac{2t\|p\|^2}{1-2s\|p\|^2}\right)^{-0.5}\frac{1}{\sqrt{1-2s\|p\|^2}}p\\ &= \left(1-2s\|p\|^2-2t\|p\|^2\right)^{-0.5}p\\ &= \psi(t+s,p).\end{aligned}$$

Finally, $\psi(0,p)=(1-0)^{-0.5}p=p$.

(b) Recall the relationship between a flow and a vector field. The flow is the set of integral curves of the vector field, and conversely given a point p we get a curve $\gamma(t)=\psi(t,p)$ with $\psi(0,p)=p$, so the tangent vector at p is $[\gamma]$. When we are in Euclidean space, we identify $[\gamma]$ with $\gamma'(0)$. So in this situation $F(p)=\partial_t\psi(t,p)|_{t=0}$.

$$\begin{aligned}F(p) &= \frac{\partial}{\partial}\bigg|_{t=0}\psi(t,p) = -0.5(1-2\|p\|^2\,t)^{-1.5}(-2\|p\|^2)p\bigg|_{t=0}\\ &= \|p\|^2\,p.\end{aligned}$$

The flow ψ is a maximal flow because if we add any additional point (s,p) to W with $2\|p\|^2 s>1$, then necessarily $(t,p)\in W$ for all $(-\infty,s)$. But then this includes the point where $2\|p\|^2 t=1$ and ψ is cannot be extended to such points. So the maximal flow of F is not defined on $\mathbb{R}\times\mathbb{R}^2$, it is not a global flow, and by Definition 2.13 we say that F is not complete.

A way to think about the completeness of a vector field without talking about its flow, is that a vector field is complete when every point has an integral curve that exists for all time. If we consider the vector field $F(p)=\|p\|^2\,p$, we see immediately that the direction of the integral curve γ is constant:

$$(\hat{\gamma})' = -\|\gamma\|^{-3}(\gamma'\cdot\gamma)\gamma + \|\gamma\|^{-1}\gamma' = -\|\gamma\|^{-3}(\|\gamma\|^4)\gamma + \|\gamma\|^{-1}(\|\gamma\|^2\gamma) = 0.$$

Therefore, for the initial point $(1,0)$ we have $\gamma'(0)=F(1,0)=(1,0)$, so $\gamma(t)=(x(t),0)$. The differential equation reduces to $x'(t)=x(t)^3$ with $x(0)=1$. This has the solution $x(t)=(1-2t)^{-0.5}$, which only exists up until time $t=0.5$. This demonstrates that there is an integral curve that does not exist for all time, so F cannot be complete.

Think why this phrasing in terms of integral curves is exactly equivalent to the statement with flows.

55. Commuting flows.

Let $a, b, c \in \mathbb{R}$ be constants and the vector fields $F, G \in \mathrm{Vec}^\infty(\mathbb{R}^3)$ be given by

$$F(x_1, x_2, x_3) = (1, x_3, -x_2) \quad \text{and} \quad G(x_1, x_2, x_3) = (a, b, c)\,.$$

(a) Determine the flows ψ_F and ψ_G induced by F and G respectively, and determine for which values of a, b, c the flows commute with one another: i.e. for all $t, s \in \mathbb{R}$

$$\psi_F(t, \psi_G(s, x)) = \psi_G(s, \psi_F(t, x)).$$

(b) Calculate $[F, G]$, and determine for which values of a, b, c the Lie bracket is zero, $[F, G] = 0$.

Solution.

(a) There is a subtle difference between local and global flows, but here we will see that these vector fields are *complete* and so generate global flows. A global flow on X is a map $\psi : \mathbb{R} \times X \to X$ with the 'initial' property $\psi(0, x) = x$ and the 'continuation' property $\psi(t, \psi(s, x)) = \psi(s + t, x)$. If we fix a point x_0 and consider where this point moves as time t changes, we get a path $\alpha_{x_0}(t) := \psi(t, x_0)$. The vector field associated to a flow is $F(x) = [\alpha_x]$. Reversing this, finding the flow associated to a vector field, requires solving a differential equation.

For F, this is the differential equation

$$x_1'(t) = 1, \;\; x_2'(t) = x_3(t), \;\; x_3'(t) = -x_2(t),$$

with initial condition $x(0) = (x_{10}), x_{20}, x_{30})$. Immediately we have $x_1(t) = t + x_{10}$. The other two components are a well-known system with solution $x_2(t) = x_{20} \cos t + x_{30} \sin t$ and $x_3(t) = -x_{20} \sin t + x_{30} \cos t$. The flow is

$$\psi_F(t, x) = (t + x_1,\; x_2 \cos t + x_3 \sin t,\; -x_2 \sin t + x_3 \cos t).$$

The DE system for G is very easy, everything moves in a straight line with constant speed, giving the flow

$$\psi_G(t, x) = x + (a, b, c)t.$$

Now we can compute

$$\begin{aligned}
&\psi_F(t, \psi_G(s, x)) = \psi_F(t, (x_1 + as, x_2 + bs, x_3 + cs)) \\
&\;= (t + x_1 + as, (x_2 + bs) \cos t + (x_3 + cs) \sin t, -(x_2 + bs) \sin t + (x_3 + cs) \cos t) \\
&\psi_G(s, \psi_F(t, x)) = (t + x_1 + as, x_2 \cos t + x_3 \sin t + bs, -x_2 \sin t + x_3 \cos t + cs).
\end{aligned}$$

The first components are always equal. The second components are equal when $bs \cos t + cs \sin t = bs$ and the third when $-bs \sin t + cs \cos t = cs$. We can divide out the s, and due to the linear independence of trig functions, it must be that $b = c = 0$.

Geometrically, the flow ψ_F moves points in a circles around the axis $x_2 = x_3 = 0$, while increasing their x_1 at a constant rate. The two flows commute exactly when ψ_G moves parallel to this axis.

(b) This is in Euclidean space, so we use the formulas from Exercise 22.

$$[F, G] = G'F - F'G = 0 \cdot F - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ c \\ -b \end{pmatrix}.$$

Thus the two fields commute when $b = c = 0$, exactly when the flows commute. This is a general truth: flows commute exactly when the vector fields commute (Corollary 2.21).

56. A trichotomy of integral curves.

Let X be a manifold, F a smooth vector field on X, $x_0 \in X$, and $\gamma : J \to X$ the maximal integral curve of F with $\gamma(0) = x_0$.

(a) Show there is a trichotomy: either γ is constant, or γ is injective, or γ is periodic, and these are mutually exclusive. Periodic means that $J = \mathbb{R}$, γ is non-constant, and there is a number $p > 0$ so that

$$\gamma(t + p) = \gamma(t) \quad \text{for all } \ t \in \mathbb{R}.$$

This number p is called a *period* of γ. It is not unique; for example if p is a period, so is $2p$.

Hint: Assume that γ is not constant or injective, and try to show that it is periodic.

(b) Show γ is constant exactly when $F(x_0) = 0$.

(c) Suppose that γ is periodic. Show that there is a *minimal period* $p_0 > 0$: that means p_0 is a period of γ and there are no other periods in the interval $0 < p < p_0$.

Hint: Prove this by contradiction.

(d) Suppose that γ is periodic. Show that any period is a multiple of the minimal period.

(e) Suppose that γ is periodic. Show that $\gamma|_{[0,p_0)}$ is injective and the map $f : \mathbb{S}^1 \to X$ defined by

$$f(\cos(\theta), \sin(\theta)) = \gamma\left(\frac{p_0}{2\pi} \cdot \theta\right) \quad \text{for all } \ \theta \in \mathbb{R}$$

is an embedding with $f[\mathbb{S}^1] = \gamma[\mathbb{R}]$. It follows that that the image $\gamma[\mathbb{R}]$ is a submanifold of X.

Hint: Constant Rank Theorem.

Solution.

(a) Clearly if γ is constant or periodic then it is not injective, and conversely if γ is injective then it is not constant or periodic. Periodic functions are by definition not constant. Therefore the three types are mutually exclusive.

Suppose now that γ is not constant or injective. Then there exists times such that $\gamma(t_0) = \gamma(t_1) = x_1$. Suppose that $t_0 < t_1$ without loss of generality. Now we apply the uniqueness of integral curves, Theorem 2.5(ii). Let $p = t_1 - t_0$ and $\alpha(t) = \gamma(t+p)$, which is still an integral curve of F and has $\alpha(t_0) = \gamma(t_1)$. Then $\alpha(t) = \gamma(t)$ for all t for which they are both defined.

In particular, because J is an open interval γ is defined for at least $[t_0, t_1] = [t_0, t_0+p]$ and α for at least $[t_0 - p, t_0]$. But then

$$\tilde{\gamma} : t \mapsto \begin{cases} \gamma(t) & \text{for } t \in [t_0, t_0 + p] \\ \alpha(t) & \text{for } t \in [t_0 - p, t_0] \end{cases}$$

is an integral curve of F with $\tilde{\gamma}(t_0) = \gamma(t_0)$. Since γ is maximal, it must be that in fact it is defined on at least $[t_0 - p, t_0 + p]$. On the other hand, α must also be a maximal integral curve, and $\tilde{\gamma}$ shows it is also defined on at least $[t_0 - p, t_0 + p]$. From the definition of α, γ must be defined on at least $[t_0 - p, t_0 + 2p]$. Every time that we iterate this argument, we show that the domain of γ extends $-p$ and $+p$ further than we had assumed. The only possibility is that $J = \mathbb{R}$. Finally then we have shown that $\gamma(t + p) = \gamma(t)$ for all $t \in \mathbb{R}$; it is periodic.

(b) If γ is constant, then $[\gamma]$ is the zero vector and so from the integral curve equation $[\gamma(t)] = F(\gamma(t))$ we have that $F(x_0) = 0$.

Conversely, if $F(x_0) = 0$ then the curve $\gamma(t) = x_0$ solves the integral curve equation. The solutions are unique.

(c) Let P be the set of positive periods. Suppose there were no minimal period. Because P is bounded from below by 0, it has an infimum $p = \inf P$. If $p > 0$, choose a sequence $p_k \in P$ converging to the infimum. Then by the continuity of γ

$$\gamma(t + p) = \lim_{k \to \infty} \gamma(t + p_k) = \lim_{k \to \infty} \gamma(t) = \gamma(t).$$

This contradicts the fact that P has no minimum. It must be that if there is no minimal period then $\inf P = 0$.

Now we continue the argument in local coordinates. Choose a chart ϕ containing $x_0 = \gamma(0)$ and consider the curve $\tilde{\gamma} : \mathbb{R} \to \mathbb{R}^n$, $\tilde{\gamma} = \phi \circ \gamma$. Again, take a sequence of periods p_k, this time which converge to zero. We compute the derivative of $\tilde{\gamma}$ at $t = 0$, using the fact that we know it exists (γ is smooth) and the equivalence between limits of functions and limits of sequences of function values:

$$\tilde{\gamma}'(0) = \lim_{k\to\infty} \frac{\tilde{\gamma}(0+p_k) - \tilde{\gamma}(0)}{p_k - 0} = \lim_{k\to\infty} \frac{\tilde{\gamma}(0) - \tilde{\gamma}(0)}{p_k} = 0.$$

Thus $F(x_0) = 0$ and it follows from the previous question that γ is constant. But this contradicts the definition of periodic. Therefore there must exist a minimal period.

(d) Let p_0 be the minimal period and p be any other period. Then so is $p + kp_0$ for any integer $k \in \mathbb{Z}$. Thus there is a unique period $p + kp_0 \in [0, p_0)$. But the only period in this interval is 0. Therefore $p = -kp_0$.

(e) If $\gamma|_{[0,p_0)}$ were not injective, then there would be points $t_0, t_1 \in [0, p_0)$ with $\gamma(t_0) = \gamma(t_1)$. We have seen in the proof of (a) that if $\gamma(t_0) = \gamma(t_1)$ then $t_1 - t_0$ is a period. So we would have a period $0 < |t_1 - t_0| < p_0$, which is a contradiction.

Consider the function $f : \mathbb{S}^1 \to X$. It is well defined because γ is periodic. Again from (b) we know that γ has non-vanishing derivative, so f is an immersion. And we have just seen that $\gamma|_{[0,p_0)}$ is injective, so f must be too. It remains to show that f is a homeomorphism, specifically that the inverse is continuous.

Choose any point x_1 of $\gamma[\mathbb{R}]$. Since f has constant rank, we know from the constant rank theorem (alternative version) that there are charts (ϕ, U) of $\mathbb{S}^1$ and (ψ, V) of X with $x_1 \in U$ so that

$$\psi \circ f \circ \phi^{-1}(\theta) = (\theta, 0, \ldots, 0).$$

Let $\Pi(x_1, x_2, \ldots, x_n) = x_1$. Then $\phi^{-1} \circ \Pi \circ \psi$ is continuous, and is equal to f^{-1} on $\gamma[\mathbb{R}] \cap V$.

Another proof uses that f is an injective immersion. Then because $\mathbb{S}^1$ is compact, by a previous exercise it is an embedding.

Additional Exercises

57. The integral curves of vector fields with the form λF.

Let X be a manifold, $F \in \mathrm{Vec}^\infty(X)$ a vector field, $\lambda \in C^\infty(X, \mathbb{R})$ a smooth function, $G := \lambda F \in \mathrm{Vec}^\infty(X)$ the rescaling of F, and $p_0 \in X$ a point.

Suppose that $\alpha : I \to X$ is an integral curve of F with $\alpha(0) = p_0$ and that $f : J \to I$ is a solution to the initial value problem

$$f'(t) = \lambda(\alpha(f(t))) \quad \text{with} \quad f(0) = 0.$$

Show then that $\beta := \alpha \circ f : J \to X$ is an integral curve of G with $0 \in J$ and $\beta(0) = p_0$. Moreover, show that every integral curve of G can be obtained in this way.

Solution. First, $0 \in J$ because the initial condition $f(0) = 0$ means that f is defined at 0, and $\beta(0) = \alpha(f(0)) = \alpha(0) = p_0$. This leaves the main property, that β is an integral curve of G. Choose any chart ϕ containing p_0. We must show that $[\beta(t)] = G(\beta(t))$, or in other words

$$(\phi \circ \beta)'(t) = T_{\beta(t)}(\phi)\, G(\beta(t)),$$

because this is the meaning of tangent vectors in a manifold being equal. We compute both sides

$$\begin{aligned}
(\phi \circ \beta)'(t) &= (\phi \circ \alpha \circ f)'(t) = (\phi \circ \alpha)'(f(t)) \cdot f'(t) = (\phi \circ \alpha)'(f(t)) \cdot \lambda(\alpha(f(t))) \\
T_{\beta(t)}(\phi)\, G(\beta(t)) &= T_{\beta(t)}(\phi) \left[\lambda(\beta(t)) \cdot F(\beta(t))\right] = \lambda(\beta(t)) \cdot T_{\beta(t)}(\phi)\, F(\beta(t)) \\
&= \lambda(\beta(t)) \cdot T_{\alpha(f(t))}(\phi)\, F(\alpha(f(t))) = \lambda(\beta(t)) \cdot (\phi \circ \alpha)'(f(t)).
\end{aligned}$$

The last equality follows because α is an integral curve for F, so $(\phi \circ \alpha)'(s) = T_{\alpha(s)}(\phi)\, F(\alpha(s))$ for all s. This shows that the two sides are in fact equal, and thus β is an integral curve of G.

Conversely, suppose that we have integral curves α of F and β of G with $\alpha(0) = \beta(0) = p_0$. If $F(p_0) = 0$, then so is G and the integral curves are simply $\alpha(t) = \beta(t) = p_0$ for all t. In this case we have $f(t) = \lambda(p_0)t$, which solves the DE.

Let us assume then that $F(p_0) \neq 0$, and so there is a neighbourhood of p_0 where F is non-zero. Thus there exists an interval $I = (-\varepsilon, \varepsilon)$ so that $\alpha : I \to \alpha[I] \subset X$ is a diffeomorphism by the inverse function theorem. Likewise, we can restrict the domain of β to J so that $f := \alpha^{-1} \circ \beta : J \to I$ is a well-defined smooth map. It remains to show that f satisfies the DE. But we have already calculated that

$$(\phi \circ \beta)'(t) = (\phi \circ \alpha \circ f)'(t) = (\phi \circ \alpha)'(f(t)) \cdot f'(t)$$

and

$$[\beta(t)] = G(\beta(t)) = \lambda(\alpha(f(t))) \cdot F(\alpha(f(t))) = \lambda(\alpha(f(t))) \cdot [\alpha(f(t))],$$

so f must satisfy this DE.

58. Integral curves on the torus.

For each $a > 0$ let

$$F_a : \mathbb{S}^1 \to T\mathbb{S}^1, \ (x_0, x_1) \mapsto \big(a(-x_1, x_0), (x_0, x_1)\big)$$

be a non-vanishing smooth vector field with the maximal integral curve $\gamma_a : \mathbb{R} \to \mathbb{S}^1$ with $\gamma_a(0) = (1,0)$.

Next we consider the 2-dimensional manifold $\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$. This subset of $\mathbb{R}^2 \times \mathbb{R}^2$ is a torus, a doughnut (donut). For constants $a, b > 0$ we define the vector field

$$G_{a,b} : \mathbb{T}^2 \to T\mathbb{T}^2,\ \big((x_1,y_1),(x_2,y_2)\big) \mapsto \big(F_a(x_1,y_1)\,,\ F_b(x_2,y_2)\big)\ .$$

(a) Prove that the curve

$$\eta_{a,b} : \mathbb{R} \to \mathbb{T}^2 = T\mathbb{S}^1 \times T\mathbb{S}^1,\ t \mapsto \big(\gamma_a(t), \gamma_b(t)\big)$$

is the maximal integral curve of $G_{a,b}$ with $\eta_{a,b}(0) = \big((1,0),(1,0)\big) \in \mathbb{T}^2$.

(b) Suppose $\frac{a}{b} \in \mathbb{Q}$. Show that $\eta_{a,b}$ is periodic and determine the minimal period.

The image is a submanifold called a *torus knot.*

(c) Suppose $\frac{a}{b} \in \mathbb{R} \setminus \mathbb{Q}$. Show that $\eta_{a,b}$ is injective, but that it is not an embedding.

Remark. In this case, the image $\eta_{a,b}[\mathbb{R}]$ is in fact dense in $\mathbb{T}^2$.

Solution.

(a) $[\eta_{a,b}] = ([\gamma_a],[\gamma_b]) \in TS^1 \times TS^1$. γ_a is the integral curve of F_a on $\mathbb{S}^1$, so $[\eta_{a,b}] = (F_a, F_b) = G_{a,b}$. It is maximal because it is defined for all $\mathbb{R}$, and it starts at the given point $\eta(0) = (\gamma_a(0), \gamma_b(0)) = ((1,0),(1,0))$.

(b) Suppose that $a/b = r/s \in \mathbb{Q}$ for $r, s \in \mathbb{N}$ with no common factors. Let $p_0 = 2\pi r/a = 2\pi s/b$. This is a period, because

$$\eta_{a,b}(t + p_0) = (\gamma_a(t + 2\pi r/a), \gamma_b(t + 2\pi s/b)) = (\gamma_a(t), \gamma_b(t)) = \eta_{a,b}(t).$$

Since γ_a is not constant, neither is $\eta_{a,b}$ and thus it must be periodic.

If p is a period of $\eta_{a,b}$ then it must be a period of both components. But we know the minimal period of γ_a is $2\pi/a$ and any other period is a multiple of this. Therefore $p = 2\pi k/a$. Likewise $p = 2\pi l/b$. So $a/b = k/l$. Because we assumed r, s had no common factors, it follows that $k = nr$ and $l = ns$. Therefore p is a multiple of p_0. Since this applies to any period, p_0 must be minimal.

(c) Suppose that a/b is irrational but η is not injective, $\eta(t_0) = \eta(t_1)$. It follows that $p = t_1 - t_0$ is a period of γ_a and γ_b and so therefore a multiple $p = 2\pi k/a = 2\pi l/b$ for integers k, l. But then $a/b = k/l$ is rational. By contradiction, if a/b is irrational then η is injective.

To see that it is not an embedding, we use Exercise 54(f). Consider the sequence $t_k = 2\pi k/a$. This gives a sequence of distinct points $((1,0), s_k) = \eta_{a,b}(t_k)$, where

$s_k = \gamma_b(t_k)$. Every infinite collection of points in a compact space must have an accumulation point, so $\{s_k\} \subset S^1$ must have an accumulation point $s \in \mathbb{S}^1 \setminus \{s_k\}$. But then $((1,0), s)$ is an accumulation point of $\{\eta_{a,b}(t_k)\}$ that does not lie in $\eta_{a,b}[\mathbb{R}]$. Exercise 54(f) now tells us that this is not an embedding.

59. Aligning coordinates with a vector field.

Again let X be a manifold. Let $n := \dim(X)$ be its dimension, $x_0 \in X$ a point, and $F \in \mathrm{Vec}^\infty(X)$ a vector field with $F(x_0) \neq 0$. Show that there is a chart (U, ϕ) containing $x_0 \in U$ such that

$$T_x(\phi)^{-1}(e_1) = F(x) \quad \text{for all } x \in U.$$

Hint: Let ψ be the maximal flow of F. Then we know that ψ is defined on $(-\varepsilon, \varepsilon) \times U'$ for some $\varepsilon > 0$ and neighbourhood $U' \ni x_0$. Next choose an $(n-1)$-dimensional submanifold S of U' with $x_0 \in S$ and $F(x_0) \notin T_{x_0}S$ (explain why there must exists such an S). Finally, apply the inverse function theorem to ψ.

Solution. We follow the advice of the hint. Let ψ be the maximal flow of F. Then we know that ψ is defined on $(-\varepsilon, \varepsilon) \times U'$ for some $\varepsilon > 0$ and neighbourhood $U' \ni x_0$. Choose some chart $\phi_1 : U'' \to \mathbb{R}^n$ with $\phi_1(x_0) = 0$ and $U'' \subseteq U'$, and let us write $F_1(y) := T_{\phi_1^{-1}(y)}(\phi_1)\, F(\phi_1^{-1}(y)) : \mathbb{R}^n \to \mathbb{R}^n$ for the vector field F in these local coordinates. Consider the hyperplane $H \subset \mathbb{R}^n$ perpendicular to $F_1(0)$. There must be some neighbourhood $V \subset H$ of the origin such that $F_1(y)$ is not parallel to H for all $y \in V$. One way to see this is to consider the 'vertical component' $F_1(y) \cdot F_1(0)$ which is non-zero at the origin and therefore is non-zero on a neighbourhood of the origin.

We now push this back up to the manifold. Let $S = \phi(V)$ be the $(n-1)$-dimensional manifold. The key idea here is that S is *transverse* to the vector field, ie the vector field is not tangent to this submanifold, so when we flow this submanifold with ψ, S_t sweeps out points of X and does not just slide along itself. We want to use this motion of S_t to make a coordinate system.

Consider the function $h : \mathbb{R} \times V \to U''$, $(t, y) \mapsto \psi(t, \phi_1^{-1}(y))$ where we identify $V \subseteq H$ with subsets $\mathbb{R}^{n-1}$. This map is full rank at $(0,0)$, because the derivatives in the y direction are tangent to S and the derivative in the t-direction is $F(x_0)$ which is not tangent to S. By the inverse function theorem there is a neighbourhood U and a smooth function $\phi : U \to \mathbb{R}^n$ which is its inverse.

In fact ϕ is a chart compatible with the atlas of X. Clearly ϕ is bijection onto its image because it is the inverse function of h. This also explains why it is a homeomorphism.

Compatibility with the atlas in this follows from the fact that h and ϕ are smooth as maps between manifolds.

To explain this coordinate chart a little further, if a point $x \in X$ can be written as $\psi(t, x_1)$ with x_1 in S, then $\phi(x) = (t, \phi_1(x_1))$. You could say that we pick a point $x_1 \in S$ and then flow it with ψ for some time. The points it flows through then have the coordinates of the 'starting point' x_1 and the 'arrival time' t. Of course, x_1 is a point in the manifold, not a point in Euclidean space, so we can't use it as a coordinate directly; instead we push it down into Euclidean space with ϕ_1, where it lies in H by definition.

The desired result now follows easily, because

$$T_x(\phi)^{-1}(e_1) = T_x(h)(e_1) = \partial_t \psi(t, x) = F(x).$$