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# Analysis III
## 9. Exercise: Flows and Integral Curves

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### Preparation Exercises

**52. Examples of integral curves and flows.**

Let $F$ be a smooth vector field on $\mathbb{R}^2$ given by

$$F(x, y) = (-y, x)$$

**(a)** Find the maximal integral curves of $F$.

**(b)** Write down the maximal flow of $F$.

**(c)** Consider the restriction of $F$ to $\mathbb{S}^1$. What are the integral curves and maximal flow?

**53. A Flow on $\mathbb{S}^2$.**

Consider the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. Define a vector field $\mathbb{F} : \mathbb{S}^2 \to \mathbb{R}^3$ by

$$F(x, y, z) = (-y, x, 0).$$

**(a)** Show that $F$ is a vector field on $S^2$ (using the identification that comes from the inclusion map $\iota : \mathbb{S}^2 \to \mathbb{R}^3$).

**(b)** What are the integral curves of $F$?

**(c)** Determine the maximal flow $\psi$ of $F$.

**(d)** Let $M := \mathbb{S}^2 \setminus \{(1, 0, 0)\}$. Find an open neighbourhood $W$ of $\{0\} \times M$ in $\mathbb{R} \times M$ so that $\psi|W$ is a flow on $M$. Is $\psi|W$ a global flow on $M$?

### In Class Exercises

**54. An example of an non-complete vector field.**

Let

$$W := \{\, (t, (x, y)) \in \mathbb{R} \times \mathbb{R}^2 \,|\, 2\,(x^2 + y^2) \cdot t < 1 \,\}$$

and

$$\psi : W \to \mathbb{R}^2,\ (t, (x, y)) \mapsto \frac{1}{\sqrt{1 - 2\,(x^2 + y^2) \cdot t}} \cdot (x, y).$$

**(a)** Show that $\psi$ is a flow on $\mathbb{R}^2$.

**(b)** Determine the corresponding vector field $F \in \mathrm{Vec}^\infty(\mathbb{R}^2)$.

**(c)** Explain why $\psi$ is the maximal flow of $F$, and why $F$ is not a complete vector field.

## 55. Commuting flows.

Let $a, b, c \in \mathbb{R}$ be constants and the vector fields $F, G \in \mathrm{Vec}^\infty(\mathbb{R}^3)$ be given by

$$F(x_1, x_2, x_3) = (1, x_3, -x_2) \quad \text{and} \quad G(x_1, x_2, x_3) = (a, b, c)\,.$$

**(a)** Determine the flows $\psi_F$ and $\psi_G$ induced by $F$ and $G$ respectively, and determine for which values of $a, b, c$ the flows commute with one another: i.e. for all $t, s \in \mathbb{R}$

$$\psi_F(t, \psi_G(s, x)) = \psi_G(s, \psi_F(t, x)).$$

**(b)** Calculate $[F, G]$, and determine for which values of $a, b, c$ the Lie bracket is zero, $[F, G] = 0$.

## 56. A trichotomy of integral curves.

Let $X$ be a manifold, $F$ a smooth vector field on $X$, $x_0 \in X$, and $\gamma : J \to X$ the maximal integral curve of $F$ with $\gamma(0) = x_0$.

**(a)** Show there is a trichotomy: either $\gamma$ is constant, or $\gamma$ is injective, or $\gamma$ is periodic, and these are mutually exclusive. Periodic means that $J = \mathbb{R}$, $\gamma$ is non-constant, and there is a number $p > 0$ so that

$$\gamma(t + p) = \gamma(t) \quad \text{for all } \; t \in \mathbb{R}.$$

This number $p$ is called a *period* of $\gamma$. It is not unique; for example if $p$ is a period, so is $2p$.

Hint: Assume that $\gamma$ is not constant or injective, and try to show that it is periodic.

**(b)** Show $\gamma$ is constant exactly when $F(x_0) = 0$.

**(c)** Suppose that $\gamma$ is periodic. Show that there is a *minimal period* $p_0 > 0$: that means $p_0$ is a period of $\gamma$ and there are no other periods in the interval $0 < p < p_0$.

Hint: Prove this by contradiction.

**(d)** Suppose that $\gamma$ is periodic. Show that any period is a multiple of the minimal period.

**(e)** Suppose that $\gamma$ is periodic. Show that $\gamma|_{[0,p_0)}$ is injective and the map $f : \mathbb{S}^1 \to X$ defined by

$$f(\cos(\theta), \sin(\theta)) = \gamma\left(\frac{p_0}{2\pi} \cdot \theta\right) \quad \text{for all } \; \theta \in \mathbb{R}$$

is an embedding with $f[\mathbb{S}^1] = \gamma[\mathbb{R}]$. It follows that that the image $\gamma[\mathbb{R}]$ is a submanifold of $X$.

Hint: Constant Rank Theorem.

**(f)** Suppose that $\gamma$ is injective and $X$ is compact. We know then that $J = \mathbb{R}$. Prove that if $\gamma[\mathbb{R}]$ has an accumulation point in $X \setminus \gamma[\mathbb{R}]$ then $\gamma$ is *not* an embedding.

## Additional Exercises

**57. The integral curves of vector fields with the form $\boldsymbol{\lambda F}$.**

Let $X$ be a manifold, $F \in \mathrm{Vec}^\infty(X)$ a vector field, $\lambda \in C^\infty(X, \mathbb{R})$ a smooth function, $G := \lambda F \in \mathrm{Vec}^\infty(X)$ the rescaling of $F$, and $p_0 \in X$ a point.

Suppose that $\alpha : I \to X$ is an integral curve of $F$ with $\alpha(0) = p_0$ and that $f : J \to I$ is a solution to the initial value problem

$$f'(t) = \lambda(\alpha(f(t))) \quad \text{with} \quad f(0) = 0.$$

Show then that $\beta := \alpha \circ f : J \to X$ is an integral curve of $G$ with $0 \in J$ and $\beta(0) = p_0$.

Moreover, show that every integral curve of $G$ can be obtained in this way.

**58. Integral curves on the torus.**

For each $a > 0$ let

$$F_a : \mathbb{S}^1 \to T\mathbb{S}^1,\ (x_0, x_1) \mapsto \big(a(-x_1, x_0), (x_0, x_1)\big)$$

be a non-vanishing smooth vector field with the maximal integral curve $\gamma_a : \mathbb{R} \to \mathbb{S}^1$ with $\gamma_a(0) = (1, 0)$.

Next we consider the 2-dimensional manifold $\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$. This subset of $\mathbb{R}^2 \times \mathbb{R}^2$ is a torus, a doughnut (donut). For constants $a, b > 0$ we define the vector field

$$G_{a,b} : \mathbb{T}^2 \to T\mathbb{T}^2,\ \big((x_1, y_1), (x_2, y_2)\big) \mapsto \big(F_a(x_1, y_1)\,,\ F_b(x_2, y_2)\big)\ .$$

**(a)** Prove that the curve

$$\eta_{a,b} : \mathbb{R} \to \mathbb{T}^2 = T\mathbb{S}^1 \times T\mathbb{S}^1,\ t \mapsto \big(\gamma_a(t), \gamma_b(t)\big)$$

is the maximal integral curve of $G_{a,b}$ with $\eta_{a,b}(0) = \big((1, 0), (1, 0)\big) \in \mathbb{T}^2$.

**(b)** Suppose $\frac{a}{b} \in \mathbb{Q}$. Show that $\eta_{a,b}$ is periodic and determine the minimal period.

The image is a submanifold called a *torus knot.*

**(c)** Suppose $\frac{a}{b} \in \mathbb{R} \setminus \mathbb{Q}$. Show that $\eta_{a,b}$ is injective, but that it is not an embedding.
*Remark.* In this case, the image $\eta_{a,b}[\mathbb{R}]$ is in fact dense in $\mathbb{T}^2$.

**59. Aligning coordinates with a vector field.**

Again let $X$ be a manifold. Let $n := \dim(X)$ be its dimension, $x_0 \in X$ a point, and $F \in \mathrm{Vec}^\infty(X)$ a vector field with $F(x_0) \neq 0$. Show that there is a chart $(U, \phi)$ containing $x_0 \in U$ such that

$$T_x(\phi)^{-1}(e_1) = F(x) \quad \text{for all } x \in U.$$

Hint: Let $\psi$ be the maximal flow of $F$. Then we know that $\psi$ is defined on $(-\varepsilon, \varepsilon) \times U'$ for some $\varepsilon > 0$ and neighbourhood $U' \ni x_0$. Next choose an $(n-1)$-dimensional submanifold $S$ of $U'$ with $x_0 \in S$ and $F(x_0) \notin T_{x_0}S$ (explain why there must exists such an $S$). Finally, apply the inverse function theorem to $\psi$.